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Tree and loop amplitudes in open twistor string theory

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Abstract: We compute the one-loop gluon amplitude for the open twistor string model of Berkovits, using a symmetric form of the vertex operators. We discuss the classical solutions in various topologies and instanton sectors and the canonical quantization of the world sheet Lagrangian. We derive the N-point functions for the gluon tree and one-loop amplitudes, and calculate a general one-loop expression for the current algebra.

Keywords: Topological Strings, Conformal Field Models in String Theory, String Duality, Supersymmetric gauge theory.

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Contents

1. Introduction

Twistor string theory [1 − 3] offers an approach to formulating a QCD string. Unlike conventional string theory, the twistor string has a finite number of states. These include massless ones described by $N = 4$ super Yang Mills theory coupled to $N = 4$ conformal supergravity in four spacetime dimensions. The original theory [1] is a topological string theory, where a finite number of states arises in the usual way. The alternative formulation [2, 3] has both Yang-Mills and supergravity particles occurring in an open string sector. Here the absence of a Regge tower of states is due to the absence of operators involving momentum for massive particles. Since the twistor string theory is some N=4 Yang-Mills field theory coupled to N=4 conformal supergravity and possibly a finite number of closed string states, we infer that this field theory system is ultraviolet finite.

In [1], gluon amplitudes with ℓ loops, and with $d + 1 - \ell$ negative helicity gluons and the rest positive helicity gluons, were associated with a topological string theory with D-instanton contributions of degree d. Beyond tree level, the occurrence of conformal supergravity states is believed to modify the gluon amplitudes from Yang Mills theory. The other version of twistor string theory [2] uses a set of first order 'b,c' world sheet variables with open string boundary conditions, and world sheet instantons. The target space of both models is the supersymmetric version of twistor space, $\mathbb{CP}^{3|4}$. In [4], a path integral construction of the tree amplitudes outlined in [2], is used to compute the three gluon tree amplitude. Some n-point Yang-Mills and conformal supergraviton string tree amplitudes are derived in [3], which discusses both models and general features of loop amplitudes. Some earlier computations of conventional field theory amplitudes in a helicity basis can be found in [5]–[7] for Yang Mills trees, [8, 9] for gravity trees, and [10]–[14] for the Yang Mills loop. General features of the twistor structure of the Yang Mills loop are described in [15, 16].

In this paper, we calculate an expression for the one-loop n-gluon amplitude in Berkovits' open twistor string theory. In section 2, we review the world sheet action, establishing a convenient notation and selecting a gauge where the world sheet abelian gauge fields have been gauged to zero. Then the topology, or instanton number, resides in the boundary conditions, for an open string, or the transition functions relating different gauge patches on the world sheet, for a closed string. We discuss the classical solutions on the sphere, disk, torus and cylinder.

In section 3, we discuss the canonical quantization of the string, gauge invariance in the corresponding operator formalism and the construction of vertex operators corresponding to gluons.

In section 4, the n-point gluon open string tree amplitudes are calculated, and they match the Parke-Taylor [17, 18], up to double trace terms from the current algebra. This extends the three-point amplitudes found via a path integral in [4], and provides an explicit oscillator quantization of some of the tree amplitudes found in [3].

In section 5, the gluon open string one-loop amplitudes are described as a product of contributions from twistor fields, ghost fields, and the current algebra. We compute the n-gluon one-loop twistor field amplitude. Both the n-point tree and loop amplitudes are computed for the maximally helicity violating (MHV) amplitudes, where the instanton number takes values $1, 2$. This can be extended straightforwardly to any instanton number, leading to amplitudes with arbitrary numbers of negative and positive helicity gluons.

In section 6, the general expressions for two-, three- and four-point one-loop current algebra correlators are given for an arbitrary Lie group. They are given in terms of Weierstrass $\mathcal{P}$ and ζ functions. We expect to discuss the derivation of these expressions and their generalizations, using recursion relations, in a later paper [19].

In section 7, we combine the parts and construct the four-point MHV one-loop gluon amplitude of the open twistor string. We show how the delta function vertex operators of the twistor string lead to a form of the final integral that is a simple product of the current algebra loop and the twistor fields loop. We do not discuss here the infrared regularization of the loop amplitude, nor how the gauge group of the current algebra is ultimately determined [20].

2. The action and classical solutions

Equations of motion and boundary conditions. The world sheet action for the twistor string introduced by Berkovits can be written in the form

$$S = S_{YZ} + S_{\rm ghost} + S_G \tag{2.1}$$

where S_G represents a conformal field theory with $c = 28$ and S_{YZ} is given by

$$S_{YZ} = \int i \left[Y^{I\mu} D_\mu Z_{IS} + Y^I_\mu \epsilon^{\mu\nu} D_\nu Z_{IP} \right] g^{\frac{1}{2}} d^2 x \,, \tag{2.2}$$

with $D_\mu = \partial_\mu - iA_\mu$ and $1 \leq I \leq 8$, and the fields Z_S, Z_P and Y_μ are homogeneous coordinates in the complex projective twistor superspace $\mathbb{CP}^{3|4}$ and are world-sheet scalars, pseudo-scalars and vectors, respectively. For the action to be real, the fields must satisfy the conditions $\overline{Y^{I\mu}} = -Y^{I\mu}$ for the bosonic components ($1 \leq I \leq 4$) and $\overline{Y^{I\mu}} = Y^{I\mu}$ for the fermionic components ($5 \leq I \leq 8$), $\overline{Z_{IS}} = Z_{IS}$, $\overline{Z_{IP}} = Z_{IP}$, but we must also have $\overline{A_\mu} = -A_\mu$, i.e. A_μ has to be pure imaginary.

The action (2.1) gives rise to the equations of motion

$$D_\mu Z_S + \epsilon_\mu{}^\nu D_\nu Z_P = 0, \qquad D'_\mu Y^\mu = 0, \qquad D'_\mu \epsilon^{\mu\nu} Y_\nu = 0 \tag{2.3}$$

where $D'_\mu = (\partial_\mu + iA_\mu) g^{\frac{1}{2}}$, to the constraint,

$$Y^\mu Z_S + Y_\nu \epsilon^{\nu\mu} Z_P = 0 \,, \tag{2.4}$$

and to the end condition on the open string,

$$Y^\mu n_\mu \delta Z_S + Y_\mu \epsilon^{\mu\nu} n_\nu \delta Z_P = 0 \,, \tag{2.5}$$

where n^ν is a vector normal to the boundary. The end condition (2.5) will be satisfied if

$$Y_\mu n^\mu \cos\alpha + Y_\mu \epsilon^{\mu\nu} n_\nu \sin\alpha = 0 \,, \qquad Z_S \sin\alpha - Z_P \cos\alpha = 0 \,, \tag{2.6}$$

for some function α, varying over the boundary, and continuous up to multiples of π. The function α changes under gauge transformations, which we shall now discuss.

In the case of a Euclidean signature for the world sheet, we can write

$$Y^\mu D_\mu Z_S + Y_\mu \epsilon^{\mu\nu} D_\nu Z_P = Y^z D_z \tilde{Z} + Y^{\bar{z}} D_{\bar{z}} Z \,, \tag{2.7}$$

where $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $Z = Z_S - iZ_P$, $\tilde{Z} = Z_S + iZ_P$, $D_z = \partial_z - iA_z$, $A_z = \frac{1}{2}(A_1 - iA_2)$, *etc.* With this notation, the equations of motion become

$$D_{\bar{z}} Z = D_z \tilde{Z} = 0, \qquad D'_z Y^z = D'_{\bar{z}} Y^{\bar{z}} = 0, \tag{2.8}$$

together with the constraints

$$Y^{\bar{z}} Z = Y^z \tilde{Z} = 0,$$

the boundary conditions

$$\tilde{Z} = UZ, \qquad Y^z n_z = -U^{-1} Y^{\bar{z}} n_{\bar{z}}, \tag{2.9}$$

where $U = e^{2i\alpha}$ in terms of (2.6), and reality conditions $\bar{Z} = \tilde{Z}$, $\overline{Y^z} = -Y^{\bar{z}}$ for bosonic components, and $\overline{Y^z} = Y^{\bar{z}}$ for fermionic components, $\overline{A_z} = -A_{\bar{z}}$. The reality conditions imply that on the boundary $|U| = 1$.

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Gauge invariance. The action has two abelian gauge invariances,

$$Y^{\bar{z}} \mapsto g^{-1} Y^{\bar{z}}, \quad Z \mapsto gZ, \quad A_{\bar{z}} \mapsto A_{\bar{z}} - ig^{-1}\partial_{\bar{z}} g\,, \tag{2.10}$$

$$Y^{z} \mapsto \tilde{g}^{-1} Y^{z}, \quad \tilde{Z} \mapsto \tilde{g}\tilde{Z}, \quad A_{z} \mapsto A_{z} - i\tilde{g}^{-1}\partial_{z} \tilde{g}\,, \tag{2.11}$$

where $g = e^{\psi+i\varphi}$, $\tilde{g} = e^{-\psi+i\varphi}$, each in $GL(1,\mathbb{C})$, so that

$$A_\mu \mapsto A_\mu + \partial_\mu \varphi + \epsilon_\mu{}^\nu \partial_\nu \psi\,, \tag{2.12}$$

and φ, ψ need to be pure imaginary, i.e. $\bar{g} = \tilde{g}$ (reducing the gauge group to one copy of $GL(1,\mathbb{C})$), if the reality condition on A_μ is to be maintained. $A_{\bar{z}}, A_z$, can be thought of as components, $\mathcal{A}_{\bar{z}}, \tilde{\mathcal{A}}_z$, taken from different gauge potentials, $\mathcal{A}_\mu, \tilde{\mathcal{A}}_\mu$, associated with the transformations $g, \tilde{g}$, respectively.

The gauge invariance of the theory can be used in general to set the potential $A_\mu = 0$, with the vestige of the gauge structure residing in the boundary conditions or the gauge transformations which relate the fields on different patches, in the case of world sheets with non-trivial global topology (and in the components $\mathcal{A}_z, \tilde{\mathcal{A}}_{\bar{z}}$, that do not appear explicitly in the action). In a gauge with $A_z = A_{\bar{z}} = 0$, the equations of motion for $Z, \tilde{Z}$ become $\partial_{\bar{z}} Z = \partial_z \tilde{Z} = 0$, i.e. $Z \equiv Z(z), \tilde{Z} \equiv \tilde{Z}(\bar{z})$.

Solutions on the sphere. If the world sheet is the sphere S^2, corresponding to closed string boundary conditions, mapped stereographically onto the plane, the potentials are defined by functions on two patches, $A_\mu^>$ on $S_\mu^> = \{z : |z| > 1 - \epsilon\}$ and $A_\mu^<$ on $S_\mu^< = \{z : |z| < 1 + \epsilon\}$, for some $\epsilon > 0$, with

$$A_{\bar{z}}^> - A_{\bar{z}}^< = -ig^{-1}\partial_{\bar{z}} g, \qquad A_z^> - A_z^< = -i\tilde{g}^{-1}\partial_z \tilde{g} \qquad \text{for } 1+\epsilon > |z| > 1-\epsilon. \tag{2.13}$$

We can apply gauge transformations $\gamma^>, \tilde{\gamma}^>, \gamma^<, \tilde{\gamma}^<$, on the two patches separately which map all of $A_{\bar{z}}^>, A_z^>, A_{\bar{z}}^<, A_z^<$ to zero, so that the gauge transformations relating the patches become

$$h = \gamma^> g (\gamma^<)^{-1}, \quad \tilde{h} = \tilde{\gamma}^> \tilde{g} (\tilde{\gamma}^<)^{-1}, \quad \text{so that} \quad \partial_{\bar{z}} h = \partial_z \tilde{h} = 0$$

implying $h \equiv h(z), \tilde{h} \equiv \tilde{h}(\bar{z})$. Maintaining the reality conditions implies that $\bar{h} = \tilde{h}$. As z encircles the unit circle $|z| = 1$, the phase of $h(z)$ will increase by $-2\pi n$ for some integer n, so that $\log(z^n h(z))$ is single-valued in the annulus $1 + \epsilon > |z| > 1 - \epsilon$. By writing $\log(z^n h(z))$ as the sum of two functions, one regular at the origin and the other at infinity, we can obtain gauge transformations on the two patches that will maintain $A_{\bar{z}}^> = A_{\bar{z}}^< = 0$ while replacing the transition function $h(z)$ by z^{-n}. Correspondingly, $\tilde{h}(\bar{z})$ is replaced by $\bar{z}^{-n}$. Thus, for the sphere, we can always choose a gauge in which the components of the potential occurring in the equations of motion are zero and the $Z, \tilde{Z}$ fields on the two patches are related by

$$Z^>(z) = z^{-n} Z^<(z), \qquad \tilde{Z}^>(\bar{z}) = \bar{z}^{-n} \tilde{Z}^<(\bar{z}). \tag{2.14}$$

(See appendix A.) These conditions have a solution provided that $n \geq 0$, in which case $Z^<(z), \tilde{Z}^<(\bar{z})$ are polynomials of order n which are complex conjugates of one another.

Solutions on the disk. If the world sheet is the disk, $D^2 = \{z : |z| \leq 1\}$, appropriate to open string tree amplitudes, we can again choose a gauge in which $A_z = A_{\bar{z}} = 0$, so that Z and $\tilde{Z}$ are analytic functions of z and $\bar{z}$, respectively, and the residue of the gauge structure is only left in the boundary conditions (2.9). If the phase of $\mathrm{U}(z)$ in this equation changes by $-2\pi n$, n an integer, as z goes round the unit circle, we can write $\log(z^n\,\mathrm{U}(z)) = f_<(z) + f_>(z)$ on the unit circle, where $f_>(z), f_<(z)$ are defined and holomorphic in $|z| \geq 1$ and $|z| \leq 1$, respectively. The fact that $|U| = 1$ on the unit circle implies that $\overline{f_<(z)} = -f_>(1/\bar{z})$. If we now apply the gauge transformation $\gamma = e^{-f_>(1/\bar{z})}, \tilde{\gamma} = e^{f_<(z)}$, $U \mapsto \gamma U \tilde{\gamma}^{-1} = z^{-n}$ on $|z| = 1$. The boundary condition only has non-trivial solutions for $n \geq 0$ and the general solution satisfying the reality and boundary conditions is then

$$Z(z) = \sum_{m=0}^{n} Z_m z^m, \qquad \tilde{Z}(\bar{z}) = \sum_{m=0}^{n} \bar{Z}_m \bar{z}^m, \tag{2.15}$$

where $Z_m = \bar{Z}_{n-m}$.

Solutions on the torus. If the world sheet is a torus, T^2, corresponding to a closed string loop amplitude, we can describe it by identifying points of the complex plane related by translations of the form $z \mapsto z + m_1 + n_1\tau$, $m_1, n_1 \in \mathbb{Z}$, for a given modulus $\tau \in \mathbb{C}$. The various copies of the fundamental region $\{z = x + y\tau : 0 \leq x, y < 1\}$ have to be related by gauge transformations, $g_a, \tilde{g}_a$,

$$A_{\bar{z}}(z+a) = A_{\bar{z}}(z) - i g_a^{-1}\partial_{\bar{z}} g_a, \quad A_z(z+a) = A_z(z) - i\tilde{g}_a^{-1}\partial_z \tilde{g}_a, \tag{2.16}$$

where $a = m_1 + n_1\tau$, $m_1, n_1 \in \mathbb{Z}$. The gauge transformations have to satisfy

$$g_{a+b}(z) = g_a(z+b) g_b(z) = g_b(z+a) g_b(z), \tag{2.17}$$

and similarly for $\tilde{g}_a$, and the reality condition $\overline{g_a(z)} = \tilde{g}_a(\bar{z})$.

We can apply gauge transformations $\gamma, \tilde{\gamma}$ to $\mathcal{A}, \tilde{\mathcal{A}}$ to set $A_z = A_{\bar{z}} = 0$ over the plane, changing the gauge transition functions $g_a(z) \mapsto h_a = \gamma(z+a) g_a(z)\gamma(z)^{-1}$, $\tilde{g}_a(z) \mapsto \tilde{h}_a = \tilde{\gamma}(z+a)\tilde{g}_a(z)\tilde{\gamma}(z)^{-1}$, where $h_a, \tilde{h}_a$ are holomorphic functions of $z, \bar{z}$, respectively. In this gauge,

$$Z(z+a) = h_a(z) Z(z), \qquad \tilde{Z}(\bar{z}+\bar{a}) = \tilde{h}_a(\bar{z})\tilde{Z}(\bar{z}). \tag{2.18}$$

Writing $h_a(z) = e^{i\rho_a(z)}$, (2.17) implies

$$\rho_1(z+\tau) - \rho_1(z) - \rho_\tau(z+1) + \rho_\tau(z) = -2\pi n, \tag{2.19}$$

for some integer n, which describes the topology of the solution. A particular gauge transformation that possesses this property and, more generally, satisfies (2.17), is

$$h_a^0(z) = \exp\left(\frac{-\pi n(a-\bar{a})}{\mathrm{Im}\tau}\left(z + \frac{a}{2}\right) + i\pi n m_1 n_1 + i\eta_a\right) \tag{2.20}$$

where $\eta_{a+b} = \eta_a + \eta_b$. In fact, if $h_a(z)$ satisfies (2.19), we would need to make a further gauge transformation to bring it into the standard form (2.20).

If we let $\eta_a = \pi m_1 \epsilon - \pi n_1 \epsilon'$, the translation property of Z is

$$Z(z+1) = e^{i\pi\epsilon} Z(z), \qquad Z(z+\tau) = e^{-i\pi(\epsilon' + n(2z+\tau))} Z(z), \tag{2.21}$$

which are the defining relations for an n-th order theta function with characteristics ϵ, ϵ', and we must have $n > 0$ for non-trivial solutions. If the usual theta function is denoted by

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\nu, \tau) = \sum_{m \in \mathbb{Z}} \exp \left\{ i\pi (m + \tfrac{1}{2}\epsilon)^2 \tau + 2\pi i (m + \tfrac{1}{2}\epsilon)\nu + \pi i m \epsilon' + \tfrac{1}{2}\pi i \epsilon \epsilon' \right\}, \tag{2.22}$$

the space of n-th order theta functions is spanned by the n functions

$$\theta \begin{bmatrix} \frac{1}{n}(\epsilon + 2p) \\ \epsilon' \end{bmatrix} (nz, n\tau), \quad p = 0, 1, \ldots n-1. \tag{2.23}$$

Thus each of the components of Z has an expansion of the form

$$Z^I(z) = \sum_{p=0}^{n-1} c_p^I \, \theta \begin{bmatrix} \frac{1}{n}(\epsilon + 2p) \\ \epsilon' \end{bmatrix} (nz, n\tau), \tag{2.24}$$

where $1 \leq I \leq 8$.

Solutions on the cylinder. For an open string loop amplitude, we need to consider the world sheet being a cylinder, C^2, which we take to be the strip region $\{z : 0 \leq \text{Re } z \leq \frac{1}{2}\}$ with z identified with $z + n\tau$, where τ is pure imaginary. The equations (2.16) and (2.17) hold but with a, b restricted to be integral multiples of τ. The fields $Z(z), \tilde{Z}(\bar{z})$ have to satisfy boundary conditions on Re $z = 0, \frac{1}{2}$,

$$\tilde{Z}(-iy) = U_0(y) Z(iy), \qquad \tilde{Z}(\tfrac{1}{2} - iy) = U_{\frac{1}{2}}(y) Z(\tfrac{1}{2} + iy), \qquad |U_0(y)| = |U_{\frac{1}{2}}(y)| = 1,$$

for real y.

We can again work in a gauge in which $A_z = A_{\bar{z}} = 0$, where $Z, \tilde{Z}$ are analytic functions of $z, \bar{z}$, respectively. We can find the gauge transformation $\gamma_0 = e^{i\rho_0}$ to take us to such a gauge by solving the equation $\mathcal{A}_{\bar{z}} = -\partial_{\bar{z}} \rho_0$ and we can impose the boundary condition $\rho_0 = \frac{1}{2}\varphi_0$ on Re $z = 0$, where $U_0(y) = e^{i\varphi_0(y)}$ with $\varphi_0(y) \in \mathbb{R}$. Under the gauge transformation, $Z \mapsto \gamma_0 Z$, $\tilde{Z} \mapsto \bar{\gamma}_0 \tilde{Z}$, so $U_0(y) \mapsto \bar{\gamma}_0(iy) U_0(y) \gamma_0(iy)^{-1} = 1$ for real y. So we can choose a gauge in which $A_z = A_{\bar{z}} = 0$ and $Z, \tilde{Z}$ are analytic functions of $z, \bar{z}$, respectively, and $Z, \tilde{Z}$ are real on Re $z = 0$. In this gauge we can extend the definition of the fields into the region $\{z : -\frac{1}{2} \leq \text{Re } z \leq 0\}$ by reflection about Re $z = 0$: $Z(z) = \overline{Z(-\bar{z})} = \tilde{Z}(-z)$, $\tilde{\mathcal{A}}_z(z, \bar{z}) = -\overline{\mathcal{A}_{\bar{z}}(-\bar{z}, -z)} = \tilde{\mathcal{A}}_z(-\bar{z}, -z)$, thus obtaining fields defined smoothly in the whole region $\{z : -\frac{1}{2} \leq \text{Re } z \leq \frac{1}{2}\}$.

Similarly we can define another gauge in which $A_z = A_{\bar{z}} = 0$ and $Z(z), \tilde{Z}(\bar{z})$ are analytic functions real on Re $z = \frac{1}{2}$ by making a gauge transformation which maps $U_{\frac{1}{2}} \mapsto 1$. In this gauge we may extend the fields over the whole strip $\{z : 0 \leq \text{Re } z \leq 1\}$ by reflection about Re $z = \frac{1}{2}$. If $Z', \tilde{Z}'$ denote the fields in this second gauge, $Z'(z) = \overline{Z'(1 - \bar{z})}$. If $\gamma, \tilde{\gamma}$ are the gauge transformation that relate the two gauges we have constructed, $Z'(z+1) = \overline{Z'(-\bar{z})} = \overline{\gamma(-\bar{z}) Z(-\bar{z})} = \overline{\gamma(-\bar{z})} Z(z)$, for $-\frac{1}{2} < \text{Re } z < 0$. So, if we assume we can extend the definition of the gauge transformation $\gamma(z)$ from $0 < \text{Re } z < \frac{1}{2}$ to $0 < \text{Re } z < 1$, we have, for $-\frac{1}{2} < \text{Re } z < 0$,

$$Z(z+1) = g_1(z)Z(z), \qquad \text{where} \quad g_1(z) = \gamma(z+1)^{-1}\overline{\gamma(-\bar{z})}.$$

Thus we have constructed from the solution defined on the cylinder, C^2, one defined on the torus, T^2, for which τ is pure imaginary, defined in a gauge in which $Z(z) = \overline{Z(-\bar{z})}$. For τ pure imaginary, the complex conjugate of (2.23), evaluated with z replaced by $-\bar{z}$, is

$$\theta \begin{bmatrix} \frac{1}{n}(\bar{\epsilon}+2p) \\ -\bar{\epsilon}' \end{bmatrix} (nz, n\tau), \quad p = 0, 1, \ldots n-1. \tag{2.25}$$

This is in the space of functions (2.24) if and only if ϵ is real and $\epsilon' = 0$ or 1. The general solution for Z for the cylinder is thus given by (2.24) with these restrictions on ϵ, ϵ'.

3. Quantization and vertices

Canonical quantization. The quantum theory involves the twistor fields $Y^I, Z^I, 1 \leq I \leq 8$, the current algebra, J^A, ghost fields, b, c, associated with the conformal invariance, and ghosts u, v, associated with the gauge invariance, and the world sheet gauge fields $\mathcal{A}_\mu, \tilde{\mathcal{A}}_\mu$ which have no kinetic term. The conformal spins and contributions of the various fields to the central charge of the Virasoro algebra are shown the following table:

	Y	Z	J^A	u	v	b	c
U(1) charge	-1	1	0	0	0	0	0
conformal spin, $\mathcal{J}$	1	0	1	1	0	2	-1
central charge, c	0		28	-2		-26	

The fields $Z^I, 1 \leq I \leq 8$, comprise four boson fields, $\lambda^a, \mu^a, 1 \leq a \leq 2$, and four fermion fields $\psi^M, 1 \leq M \leq 4$; the gauge invariance insures that the Z^I are effectively projective coordinates in the target space $\mathbb{CP}^{3|4}$. As we saw in section 2, in a gauge in which $A_z = \tilde{\mathcal{A}}_z = 0$ and $A_{\bar{z}} = \mathcal{A}_{\bar{z}} = 0$, $\lambda^a(z), \mu^a(z), \psi^M(z)$ are holomorphic, and the only further effect of the gauge fields is through their topology. Since the contribution to the Virasoro central charge c from the fermionic and bosonic twistor fields cancels to zero, and the ghost fields have $c = -26 - 2$, the current algebra J^A is required to have $c = 28$.

The mode expansion of the basic fields takes the form

$$\Phi(z) = \sum \Phi_n z^{-n-\mathcal{J}} \tag{3.1}$$

where Φ stands for Y, Z, u, v, b, c, or J^A, and $\mathcal{J}$ denotes the conformal spin of the relevant fields. The vacuum state $|0\rangle$ satisfies $\Phi_n|0\rangle = 0$ for $n > -\mathcal{J}$.The canonical commutation relations for the basic fields are

$$[\![Z^i_m, Y^j_n]\!] = \delta^{ij}\delta_{m,-n}, \quad \{c_m, b_n\} = \delta_{m,-n}, \quad \{v_m, u_n\} = \delta_{m,-n}, \tag{3.2}$$

where the brackets $[\![,]\!]$ denote commutators when either i or j is not greater than 4 and anticommutators when both i and j are greater than 4; and

$$[J^A_m, J^B_n] = if^{AB}{}_C J^C_{m+n} + km\delta_{m,-n}\delta^{AB}. \tag{3.3}$$

These lead to the normal ordering relations

$$Z^i(z)Y^j(\zeta) =: Z^i(z)Y^j(\zeta) : + \frac{\delta^{ij}}{z-\zeta}, \tag{3.4}$$

and similar relations for c, b and v, u.

The Virasoro algebra is given by

$$L(z) = -\sum_j : Y^j(z)\partial Z^j(z) : - : u(z)\partial v(z) : +2 : \partial c(z)b(z) : - : \partial b(z)c(z) : + L^J(z), \tag{3.5}$$

where the moments of $L^J(z)$ constitute the Virasoro algebra associated with the current algebra.

The BRST current is

$$Q(z) = c(z)\tilde{L}(z) + v(z)J(z) - : c(z)b(z)\partial c(z) : + \tfrac{3}{2}\partial^2 c(z), \tag{3.6}$$

where

$$\tilde{L}(z) = -\sum_j : Y^j(z)\partial Z^j(z) : - : u(z)\partial v(z) : + L^J(z).$$

$Q(z)$ is a primary conformal field of dimension one with respect to $L(z)$, and the BRST charge Q_0 satifies $Q_0^2 = 0$, where $Q(z) = \sum_n Q_n z^{-n-1}$. $L(z)$ generates a Virasoro algebra with total central charge zero.

Gauge invariance. The current associated with the gauge transformation is

$$J(z) = -P(z) = -\sum_{j=1}^{8} P^j(z), \tag{3.7}$$

where

$$P^j(z) =: Y^j(z)Z^j(z) := \sum_m a^j_m z^{-m-1} \quad \text{for each } j; \tag{3.8}$$

then

$$[a^i_m, a^j_n] = \epsilon^i \delta^{ij} m\delta_{m,-n}, \qquad [a^i_m, Y^j_n] = Y^j_{m+n}\delta^{ij}, \qquad [a^i_m, Z^j_n] = -Z^j_{m+n}\delta^{ij}, \tag{3.9}$$

where $\epsilon^i = -1$, if $i \leq 4$, and $\epsilon^i = 1$, otherwise. Hence

$$[a^i_m, Y^j(z)] = z^m Y^j(z)\delta^{ij}, \qquad [a^i_m, Z^j(z)] = -z^m Z^j(z)\delta^{ij}.$$

If we introduce $e^{q^i_0}$ so that

$$e^{q^i_0} a^j_0 e^{-q^i_0} = a^j_0 - \epsilon^i \delta^{ij},$$

where ϵ^i is as in (3.9), the normal ordered products

$$ {}^\times_\times e^{\pm X^j(z)} {}^\times_\times \equiv e^{\pm q_0^j} \exp\left\{\mp\sum_{n<0}\frac{1}{n}a_n^j z^{-n}\right\}\exp\left\{\mp\sum_{n>0}\frac{1}{n}a_n^j z^{-n}\right\} z^{\pm a_0^j}, \tag{3.10} $$

define fermion fields, which can be identified with $Y^j(z)$, $Z^j(z)$ for $j \geq 5$. In general, formally,

$$ X^j(z) = q_0^j + a_0^j \log z - \sum_{n\neq 0}\frac{1}{n}a_n^j z^{-n}. $$

Although, in the fermionic cases $j \geq 5$, we have an equivalence between the spaces generated by Y_n^j, Z_n^j, on the one hand, and $e^{\pm q_0^j}, a_n^j$, on the other,

$$ Y^j(z) = {}^\times_\times e^{X^j(z)} {}^\times_\times, \qquad Z^j(z) = {}^\times_\times e^{-X^j(z)} {}^\times_\times, \qquad j \geq 5, \tag{3.11} $$

in the bosonic cases $j \leq 4$, we need to supplement $e^{\pm q_0^j}, a_n^j$ by two fermionic fields $\xi^j(z), \eta^j(z)$,

$$ \{\xi_m^i, \eta_n^j\} = \delta^{ij}\delta_{m,n}; \qquad \eta_n|0\rangle = 0,\ n \geq 0, \quad \xi_n|0\rangle = 0,\ n \geq 1. $$

Note that in this case $\xi, \eta, e^{\pm q_0}, a$ generate a larger space than the fields Y, Z. Then we can write

$$ Y^j(z) = {}^\times_\times e^{-X^j(z)} {}^\times_\times \partial\xi^j(z), \qquad Z^j(z) = {}^\times_\times e^{X^j(z)} {}^\times_\times \eta^j(z), \qquad j \leq 4. \tag{3.12} $$

If $g(z)$ is an analytic function, representing a gauge transformation, defined in some annular neighborhood of the unit circle, $|z| = 1$, with winding number d as z goes round the unit circle,

$$ g(z) = z^d e^{-\alpha(z)}, \qquad \alpha(z) = \sum_{n=-\infty}^{\infty}\alpha_n z^{-n} = \alpha^<(z) + \alpha_0 + \alpha^>(z), \tag{3.13} $$

where

$$ \alpha^<(z) = \sum_{n>0}^{\infty}\alpha_n z^{-n}, \qquad \alpha^>(z) = \sum_{n<0}^{\infty}\alpha_n z^{-n}. \tag{3.14} $$

Defining $P(z)$ as in (3.7), with

$$ P(z) = \sum_n a_n z^{-n-1}, \qquad e^{\pm q_0} = \prod_{j=1}^{8} e^{\pm q_0^j}, $$

let

$$ \begin{aligned} P[\alpha] &= \frac{1}{2\pi i}\oint P(z)\alpha(z)dz = \sum_{-\infty}^{\infty} a_n\alpha_{-n} = P[\alpha^<] + a_0\alpha_0 + P[\alpha^>], \\ P[\alpha^<] &= \sum_{n<0}\alpha_{-n}a_n, \\ P[\alpha^>] &= \sum_{n>0}\alpha_{-n}a_n. \end{aligned} $$

So, if

$$U_g = e^{dq_0} e^{P[\alpha]} = e^{dq_0} e^{P[\alpha^<]} e^{a_0\alpha_0} e^{P[\alpha^>]}, \tag{3.15}$$

noting that the a_n commute among themselves,

$$U_g Y^j(z) U_g^{-1} = g(z)^{-1} Y^j(z), \qquad U_g Z^j(z) U_g^{-1} = g(z) Z^j(z).$$

The vertex operators $V_j(z_j)$ are gauge invariant if $[a_n, V_j(z_j)] = 0$, so that

$$e^{P[\alpha]} V_j(z_j) e^{-P[\alpha]} = V_j(z_j), \tag{3.16}$$

we have

$$\langle 0|U_g V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle = \langle 0|e^{dq_0} V_1(z_1)V_2(z_2)\dots V_n(z_n)|0\rangle, \tag{3.17}$$

showing that the winding number, the topology of the gauge transformation, is the only part affecting the amplitude. The winding number d can be regarded as labeling different instanton sectors, which one needs to sum over, described by the operator e^{dq_0}.

Because the trace

$$\begin{aligned} \mathrm{tr}\left(a_m V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0}\right) &= w^m \mathrm{tr}\left(V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0} a_m\right) \\ &= w^m \mathrm{tr}\left(a_m V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0}\right), \end{aligned}$$

we have

$$\mathrm{tr}\left(a_m V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0}\right) = 0, \qquad \text{if } m \neq 0,$$

so that

$$\mathrm{tr}\left(U_g V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0}\right) = \mathrm{tr}\left(e^{dq_0} u^{a_0} V_1(z_1)V_2(z_2)\dots V_n(z_n) w^{L_0}\right), \tag{3.18}$$

showing that the trace depends on the instanton number d and $u^{a_0} = e^{\alpha_0 a_0}$. It is necessary to sum over d and average over $u = e^{\alpha_0}$ with respect to the invariant measure $du/u = d\alpha_0$. The evaluation of such traces is discussed in appendix C.

Scalar products. The reality conditions on Y^I, Z^J imply that

$$\left(Z_n^J\right)^\dagger = Z_{-n}^J, \tag{3.19}$$

$$(Y_n^J)^\dagger = -Y_{-n}^J \quad \text{for} \quad 1 \leq J \leq 4, \qquad (Y_n^J)^\dagger = Y_{-n}^J \quad \text{for} \quad 5 \leq J \leq 8. \tag{3.20}$$

It follows from (3.11) and (3.12) that

$$Y_{n-d}^I e^{dq_0} = e^{dq_0} Y_n^I, \qquad Z_{n+d}^I e^{dq_0} = e^{dq_0} Z_n^I \qquad \text{for } 1 \leq I \leq 8. \tag{3.21}$$

The modes Y_n^I, Z_n^I satisfy the vacuum conditions

$$Y_n^I|0\rangle = 0, \quad n \geq 0, \qquad Z_n^I|0\rangle = 0, \quad n \geq 1. \tag{3.22}$$

If we take a single *fermionic* component of Y^I, Z^I in isolation (i.e. $5 \leq I \leq 8$), denoted Y, Z, scalar products can be evaluated from the basic relation $\langle 0|Z_0|0\rangle = 1$. With e^{dq_0}

defined for this component similarly to the above for integral d, we can show that $Z_0|0\rangle = e^{-q_0}|0\rangle$, and, more generally, $Z_{1-d}\dots Z_0|0\rangle = e^{-dq_0}|0\rangle$, for positive integers d, so that

$$\langle 0|e^{dq_0} Z_{-d}\dots Z_0|0\rangle = 1, \tag{3.23}$$

and $\langle 0|e^{dq_0} Z_{-n_1}\dots Z_{-n_m}|0\rangle = 0$ for other products $Z_{-n_1}\dots Z_{-n_m}$ (unless $m = d+1$ and the $n_1, \dots, n_{d+1}$ are a permutation of $0, \dots, d$).

For a single *bosonic* component of Y^I, Z^I in isolation (i.e. $1 \leq I \leq 4$), again denoted Y, Z, scalar products can be evaluated from the basic relation

$$\langle 0|f(Z_0)|0\rangle = \int f(Z_0)dZ_0, \qquad \text{or, equivalently,} \qquad \langle 0|e^{ikZ_0}|0\rangle = \delta(k). \tag{3.24}$$

In the bosonic case, the matrix elements of e^{q_0} are specified by

$$\langle 0|e^{ik'Z_0} e^{q_0} e^{ikZ_0}|0\rangle = \langle 0|e^{q_0} e^{ik'Z_{-1}} e^{ikZ_0}|0\rangle = \delta(k')\delta(k) \tag{3.25}$$

and, analogously to (3.23),

$$\langle 0|e^{dq_0} \exp\left\{ i\sum_{j=0}^{d} k_j Z_{-j} \right\} |0\rangle = \prod_{j=0}^{d} \delta(k_j), \tag{3.26}$$

equivalently,

$$\langle 0|e^{dq_0} f(Z_0, \dots, Z_{-d})|0\rangle = \int f(Z_0, \dots, Z_{-d}) dZ_0, \dots, dZ_{-d}. \tag{3.27}$$

[Alternatively, instead of including the factor e^{dq_0}, we could calculate tree diagrams working in a twisted vacuum $|\hat{0}\rangle = e^{\frac{1}{2}dq_0}|0\rangle$, where Y, Z have modified conformal dimensions $1 + \frac{1}{2}d$ and $-\frac{1}{2}d$, respectively. Here we will take the former approach as a basis for constructing loop contributions.]

In the cases at hand, with all eight components of Y^I, Z^I present, the expressions have an overall scaling invariance under $Z^I(z) \mapsto kZ^I(z), Y^I(z) \mapsto k^{-1}Y^I(z)$, which would cause the integral over $Z_0^I, 1 \leq I \leq 4$, to diverge, unless the invariant measure on this group is divided out to leave the invariant measure on the coset space. This invariant measure can be taken to be $d\gamma_S = dZ^j/Z^j$, with the various choices of j giving equivalent answers and vacuum expectation value has the form $\underline{Z}$

$$\langle 0|f(\underline{Z}_0)|0\rangle = \int f(\underline{Z}_0) d^4\underline{Z}_0/d\gamma_S = \int f(\underline{Z}_0) Z_0^{j_0} d^4\underline{Z}_0/dZ_0^{j_0} \tag{3.28}$$

where $\underline{Z}_0 = (Z_0^1, Z_0^2, Z_0^3, Z_0^4)$, for any choice of j_0. [Note that, before the vacuum expectation values are taken on the fermionic modes, $Z_0^I, 5 \leq I \leq 8$, the integrand is a homogeneous function of all the Z^I, but after this has been done, the residual function $f(\underline{Z}_0)$ has the homogeneity property $f(k\underline{Z}_0) = k^{-4}f(\underline{Z}_0)$].

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Vertices. The target space of the string theory is the twistor superspace $\mathbb{CP}^{3|4}$. We use

$$Z = \begin{pmatrix} \lambda \\ \mu \\ \psi^\lambda \\ \psi^\mu \end{pmatrix}, \quad \psi^\lambda = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad \psi^\mu = \begin{pmatrix} \psi^3 \\ \psi^4 \end{pmatrix}, \tag{3.29}$$

where the $\psi^I, 1 \leq I \leq 4$ are anticommuting quantities and $\lambda, \mu \in \mathbb{C}^2$, to denote homogeneous coordinates in this space, identifying Z and kZ for nonzero $k \in \mathbb{C}$.

The vertex operator corresponding to the physical state $|\Psi\rangle = f(Z_0) J^A_{-1}|0\rangle$ of the string, which corresponds to a gluon state, is

$$V(\Psi, z) = f(Z(z)) J^A(z). \tag{3.30}$$

The state $|\Psi\rangle$ is gauge invariant provided that $f(Z_0)$ is an homogeneous function of Z_0, $f(kZ_0) = f(Z_0)$ for nonzero $k \in \mathbb{C}$, and then $V(\Psi, z)$ satisfies (3.16). $f(Z_0)$ is the wave function describing the dependence of the state Ψ on the mean position of the string in twistor superspace. Leaving aside the need to take account of the homogeneity of the coordinates, the wave function for the string being at Z' would be $\prod_I \delta(Z^I(z) - Z'^I)$. Thus, allowing for this, the wave function for $Z(z)$ to be at $Z' = (\pi, \omega, \theta)$ is

$$W(z) = \int \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a)\delta(k\mu^a(z) - \omega^a) \prod_{b=1}^{4} \left(k\psi^b(z) - \theta^b\right) \frac{dk}{k}, \tag{3.31}$$

noting that $\delta(k\psi - \theta) = k\psi - \theta$ is the form of the delta function for anticommuting variables. [(3.31) is invariant under $Z(z) \mapsto k(z)Z(z)$ and separately under $Z' \mapsto kZ'$.]

To obtain the form of the vertex used by Berkovits and others [2]–[3], use the first delta function, $\delta(k\lambda^1(z) - \pi^1)$ to do the integration in (3.31), to obtain

$$W(z) = \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right) \prod_{a=1}^{2} \delta\left(\frac{\mu^a(z)}{\lambda^1(z)} - \frac{\omega^a}{\pi^1}\right) \prod_{b=1}^{4} \left(\frac{\psi^b(z)}{\lambda^1(z)} - \frac{\theta^b}{\pi^1}\right). \tag{3.32}$$

Then multiply by $\exp(i\omega^b\bar{\pi}_b)$, to Fourier transform on ω^b, and by

$$A(\theta) = A_+ + \theta^b A_b + \frac{1}{2}\theta^b\theta^c A_{bc} + \frac{1}{3!}\theta^b\theta^c\theta^d A_{bcd} + \theta^1\theta^2\theta^3\theta^4 A_-, \tag{3.33}$$

(so that A_{bc} corresponds to the six scalar bosons, A_b and A_{bcd} to the two helicity states of the four spin $\frac{1}{2}$ fermions, and $A_\pm$ to the two helicity states of the spin one boson, in the $(\pm 1, 4(\pm\frac{1}{2}), 6(0))$ supermultiplet of N=4 Yang Mills theory) and integrate with respect to ω, θ, yielding

$$\frac{1}{(\pi^1)^2}\delta\left(\frac{\lambda^2}{\lambda^1} - \frac{\pi^2}{\pi^1}\right) \exp\left\{i\frac{\mu^a\bar{\pi}_a\pi^1}{\lambda^1}\right\}$$
$$\times \left[A_+ + \frac{\pi^1}{\lambda^1}\psi^b A_b + \left(\frac{\pi^1}{\lambda^1}\right)^2 \frac{1}{2}\psi^b\psi^c A_{bc} + \left(\frac{\pi^1}{\lambda^1}\right)^3 \frac{1}{3!}\psi^b\psi^c\psi^d A_{bcd} + \left(\frac{\pi^1}{\lambda^1}\right)^4 \psi^1\psi^2\psi^3\psi^4 A_-\right],$$

where b, c, d are summed over. This is essentially the vertex used by Berkovits, except that he omits the A_b, A_{bc}, A_{bcd} terms.

The form of the vertex that it will be convenient to use in what follows is that obtained from (3.31) by Fourier transforming on ω^a and multiplying by $A(\theta)$ and integrating with respect to θ^a,

$$\widehat{W}(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^b(z)\bar{\pi}_b} \\ \times \left[A_+ + k\psi^b A_b + \frac{k^2}{2}\psi^b\psi^c A_{bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d A_{bcd} + k^4\psi^1\psi^2\psi^3\psi^4 A_- \right], \quad (3.34)$$

where $\psi^b \equiv \psi^b(z)$. So, the vertices for the negative and positive helicity gluons are

$$V_-^A(z) = \int dk k^3 \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} J^A(z)\psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z) \quad (3.35)$$

and

$$V_+^A(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} J^A(z), \quad (3.36)$$

respectively.

Cohomology. The standard argument that only the states in the cohomology of Q contribute to a suitable trace over a space $\mathcal{H}$, provided that $Q^\dagger = Q$ and $Q^2 = 0$, can be phrased as follows. Assuming the scalar product is non-singular on $\mathcal{H}$, we can write $\mathcal{H} = \Phi \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}$, with the following holding: $\mathcal{N} = \mathrm{Im}Q$; $\Phi \oplus \mathcal{N} = \mathrm{Ker}\ Q$; $\mathcal{N}$, $\tilde{\mathcal{N}}$ are null and orthogonal to Φ; and $\mathcal{N} = Q\tilde{\mathcal{N}}$. The cohomology of $Q = \mathrm{Ker}\ Q/\mathrm{Im}\, Q \cong \Phi$. Taking bases $|e_m\rangle$ for Φ, $|n_i\rangle$ for $\mathcal{N}$ and $|\tilde{n}_i\rangle$ for $\tilde{\mathcal{N}}$, with $\langle e_m|e_n\rangle = \epsilon_m\delta_{mn}$, $\langle n_i|\tilde{n}_j\rangle = \delta_{ij}$, $\langle n_i|n_j\rangle = \langle\tilde{n}_i|\tilde{n}_j\rangle = \langle n_i|e_m\rangle = \langle\tilde{n}_i|e_m\rangle = 0$, then the resolution of unity is

$$1 = \sum \epsilon_m |e_m\rangle\langle e_m| + \sum |n_i\rangle\langle\tilde{n}_i| + \sum |\tilde{n}_i\rangle\langle n_i|.$$

$Q|\tilde{n}_i\rangle$ is a basis for $\mathcal{N}$, so $|n_i\rangle = M_{ij}Q|\tilde{n}_j\rangle$ and $\delta_{ik} = M_{ij}\langle\tilde{n}_k|Q|\tilde{n}_j\rangle$. Then M^T is the inverse of the matrix $\langle\tilde{n}_i|Q|\tilde{n}_j\rangle$, $\overline{M}_{ij} = M_{ji}$, and

$$1 = \sum \epsilon_m |e_m\rangle\langle e_m| + \sum Q|\tilde{n}_j\rangle M_{ij}\langle\tilde{n}_i| + \sum |\tilde{n}_j\rangle M_{ij}\langle\tilde{n}_i|Q.$$

Then if we consider a trace over $\mathcal{H}$, $\mathrm{Tr}_{\mathcal{H}}\left(A(-1)^F\right)$, of an operator A, which commutes with Q and anticommutes with $(-1)^F$,

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}\left(A(-1)^F\right) &= \sum \epsilon_m \langle e_m|A(-1)^F|e_m\rangle + \sum M_{ij}\langle\tilde{n}_i|A(-1)^F Q|\tilde{n}_j\rangle + \sum M_{ij}\langle\tilde{n}_i|QA(-1)^F|\tilde{n}_j\rangle \\ &= \sum \epsilon_m\langle e_m|A|e_m\rangle = \mathrm{Tr}_\Phi(A). \end{aligned}$$

provided that $F = 0$ on Φ.

We are interested in $Q = Q_0$, the zero mode of the BRST current (3.6), and F is the fermion number for the ghost fields, so that $(-1)^F$ does indeed anticommute with Q.

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If we consider

$$A = e^{dq_0} \int \prod_{r=1}^{n} d\rho_r V_{\epsilon_r}^{A_r}(\rho_r) w^{L_0} \tag{3.37}$$

where the vertex operators $V_\epsilon^A(\rho)$ are given by (3.36) and (3.35), Q commutes with the integral of the product vertices as consequence of their having conformal spin 1 and U(1) charge 0, but it does not commute with e^{dq_0},

$$[Q, e^{dq_0}] = d \sum_n c_{-n} a_n e^{2q_0}.$$

To correct for this, replace e^{dq_0} by $e^{dq_0} e^{-d\Sigma_n c_{-n} u_n}$ in (3.37),

$$A = e^{dq_0} e^{-d\Sigma_n c_{-n} u_n} \int \prod_{r=1}^{n} d\rho_r V_{\epsilon_r}^{A_r}(\rho_r) w^{L_0}, \tag{3.38}$$

ensuring that only states in the cohomology of Q contribute to the trace $\mathrm{tr}(A(-1)^F)$. A similar instanton number changing operator is used in a different context in [21]. The inclusion of the factor $e^{-d\Sigma_n c_{-n} u_n}$ does not change the value of this trace as we can see by expanding it in a power series.

4. Tree amplitudes

In this section we compute the N-gluon MHV twistor string tree amplitudes, first with oscillators, and then compare with the path integral method, in preparation for computing the loop amplitude, which will be given in terms of a factor times the tree amplitude. The three-point trees were derived in [4], and various N-point trees were computed in [3] using aspects of Witten's twistor string theory [1].

The n-point tree amplitude, corresponding to gluons, in instanton sector d is given by

$$\mathcal{A}_n^{\mathrm{tree}} = \int \langle 0| e^{dq_0} V_{\epsilon_1}^{A_1}(z_1) V_{\epsilon_2}^{A_2}(z_2) \ldots V_{\epsilon_n}^{A_n}(z_n)|0\rangle \prod_{r=1}^{n} dz_r \Big/ d\gamma_M d\gamma_S \tag{4.1}$$

where $d\gamma_M$ is the invariant measure on the Möbius group $d\gamma_S$ is the invariant measure on the group of scale transformations on Z (as in (3.28)), and V_ϵ^A is given in (3.36) or (3.35) as $\epsilon = \pm$. It follows directly from these expressions and from (3.23) that the n-point tree amplitude for gluons vanishes unless the number of negative helicity gluons is $d+1$.

The $d = 0$ sector. If $d = 0$, we may take $\epsilon_1 = -$ and all other $\epsilon_r = +$. In the vacuum expectation value $\langle 0|V_-^{A_1}(z_1) V_+^{A_2}(z_2) \ldots V_+^{A_n}(z_n)|0\rangle$ the $Z^I(z_r)$ can be replaced by the constant Z_0^I, because all the Z_n^I, for $n \neq 0$, can be removed by annihilation on the vacuum on either left or right. The resulting expression involves integrals over the zero modes, $\lambda_0^1, \lambda_0^2, \mu_0^1, \mu_0^2$ of Z^I, $1 \le I \le 4$, and the n scaling variables k_r associated with the n vertices as in (3.36) and (3.35).

Leaving aside the current algebra from the vacuum expectation value of the $J^{A_r}(z_r)$, and $d\gamma_M$, the amplitude is proportional to

$$\int k_1^4 \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{r=1}^{n}\prod_{a=1}^{2} \delta({\pi_r}^a - k_r\lambda^a) e^{ik_r\mu^a\bar\pi_{ra}} \prod_{a=1}^{2} d\lambda^a d\mu^a/d\gamma_S$$
$$= \int k_1^4 \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{r=1}^{n}\prod_{a=1}^{2} \delta({\pi_r}^a - k_r\lambda^a) \prod_{a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r\bar\pi_{ra}\right) \prod_{a=1}^{2} d\lambda^a/d\gamma_S \qquad (4.2)$$

involves $n+1$ integrals from $d^nkd^2\lambda/d\gamma_S$ and $2n+2$ delta functions, leaving in effect $n+1$ delta functions after the integrals have been done. For $n > 3$, this exceeds the number required for momentum conservation and the amplitude vanishes.

So for $d = 0$, we need only consider $n = 3$; then the amplitude is

$$\mathcal{A}^{\rm tree}_{-++} = \int k_1^4 \prod_{r=1}^{3} \frac{dk_r}{k_r} \prod_{r=1}^{n}\prod_{a=1}^{2} \delta({\pi_r}^a - k_r\lambda^a) f^{A_1A_2A_3} \prod_{a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r\bar\pi_{ra}\right) d\lambda^a/d\gamma_S, \qquad (4.3)$$

where the invariant measure of the Möbius group, $d\gamma_M$, which replaces the three ghost fields $\langle 0|\prod_{r=1}^3 c(z_r)|0\rangle$, has cancelled against the contribution from the tree level current algebra correlator that we have normalized as $\langle 0|J_1^A(z_1)J_2^A(z_2)J_3^A(z_3)|0\rangle = f^{A_1A_2A_3}(z_1 - z_2)^{-1}(z_2-z_3)^{-1}(z_3-z_1)^{-1}$. The current algebra arises from the $c = 28$, S_G sector of the world sheet theory. The structure constants f^{ABC} supply the non-abelian gauge group for the $D = 4, N = 4$ Yang Mills theory, as explained in [2].

Now, using the three $\delta({\pi_r}^1 - k_r\lambda^1)$ to perform the k_r integrals:

$$\mathcal{A}^{\rm tree}_{-++} = \prod_{a=1}^{2} \delta\left(\sum_{r=1}^{3} {\pi_r}^1\bar\pi_{ra}\right) \frac{({\pi_1}^1)^3}{{\pi_2}^1{\pi_3}^1} \int \prod_{r=1}^{3} \delta\left({\pi_r}^2 - {\pi_r}^1\frac{\lambda^2}{\lambda^1}\right) \frac{d\lambda^2}{\lambda^1} f^{A_1A_2A_3},$$

where we have also made the choice $d\gamma_S = d\lambda^1/\lambda^1$ in line with (3.28). Note that, alternatively, we could have used the $\delta({\pi_r}^2 - k_r\lambda^2)$ delta functions to perform the k_r integrals obtaining

$$\delta\left(\sum_{r=1}^{3} {\pi_r}^2\bar\pi_{ra}\right) \quad \text{rather than} \quad \delta\left(\sum_{r=1}^{3} {\pi_r}^1\bar\pi_{ra}\right);$$

together these four delta functions correspond to momentum conservation. To draw this out as an explicit overall factor, note that

$$\sum_{r=1}^{3} {\pi_r}^2\bar\pi_{ra} = \sum_{r=1}^{3}\left({\pi_r}^2 - {\pi_r}^1\frac{\lambda^2}{\lambda^1}\right)\bar\pi_{ra} \quad \text{when} \quad \sum_{r=1}^{3} {\pi_r}^1\bar\pi_{ra} = 0$$

and the Jacobian to change between

$$\prod_{r=2,3} \delta\left({\pi_r}^2 - {\pi_r}^1\frac{\lambda^2}{\lambda^1}\right) \quad \text{and} \quad \prod_{a=1,2} \delta\left(\sum_{r=1}^{3}\left({\pi_r}^2 - {\pi_r}^1\frac{\lambda^2}{\lambda^1}\right)\bar\pi_{ra}\right)$$

is $\bar\pi_{21}\bar\pi_{32} - \bar\pi_{22}\bar\pi_{31} = [2,3]$.

So, writing

$$\prod_{a,b} \delta \left(\sum_{r=1}^{3} {\pi_r}^b \bar{\pi}_{ra} \right) \equiv \delta^4(\Sigma \pi_r \bar{\pi}_r),$$

$$\begin{aligned} \mathcal{A}^{\text{tree}}_{-++} &= \delta^4(\Sigma\pi_r\bar{\pi}_r)[2,3] \int \frac{({\pi_1}^1)^3}{{\pi_2}^1 {\pi_3}^1} \delta \left({\pi_1}^2 - {\pi_1}^1 \frac{\lambda^2}{\lambda^1} \right) \frac{d\lambda^2}{\lambda^1} f^{A_1A_2A_3} \\ &= \delta^4(\Sigma\pi_r\bar{\pi}_r)[2,3] \frac{({\pi_1}^1)^2}{{\pi_2}^1 {\pi_3}^1} f^{A_1A_2A_3} \\ &= \delta^4(\Sigma\pi_r\bar{\pi}_r) \frac{[2,3]^3}{[1,2][3,1]} f^{A_1A_2A_3}. \end{aligned} \tag{4.4}$$

Writing the general polarization vector, ϵ_r, of the r-th gluon, as the sum of positive and negative helicity parts, $\epsilon_r = \epsilon_r^+ + \epsilon_r^-$. These parts can be normalized by choosing vectors $s_{r\dot{a}}$ and $\bar{s}_{ra}$ are defined such that ${\pi_r}^a \bar{s}_{ra} = 1$ and $\bar{\pi}_r^{\dot{a}} s_{r\dot{a}} = 1$, for each r, and setting

$$\epsilon_r^+ = A_r^+ \bar{s}_{ra} \bar{\pi}_{r\dot{a}}, \qquad \epsilon_r^- = A_r^- \pi_{ra} s_{r\dot{a}}. \tag{4.5}$$

Multiplying the appropriate amplitudes $A_r^\pm$ onto $\mathcal{A}^{\text{tree}}_{-++}$, we obtain

$$\begin{aligned} \hat{\mathcal{A}}^{\text{tree}}_{-++} &= \delta^4(\Sigma\pi_r\bar{\pi}_r) \frac{[2,3]^3}{[1,2][3,1]} f^{A_1A_2A_3} A_1^- A_2^+ A_3^+ \\ &= \delta^4(\pi_r\bar{\pi}_r) f^{A_1A_2A_3} \left(\epsilon_1^- \cdot \epsilon_2^+ \epsilon_3^+ \cdot p_1 + \epsilon_2^+ \cdot \epsilon_3^+ \epsilon_1^- \cdot p_2 + \epsilon_3^+ \cdot \epsilon_1^- \epsilon_2^+ \cdot p_3 \right), \end{aligned} \tag{4.6}$$

as shown in appendix B.

The $d = 1$ sector. The one-instanton sector contributes to all gluon tree amplitudes with just two negative helicities. For $d = 1$, we have to evaluate

$$\langle 0 | e^{q_0} V_-^{A_1}(z_1) V_-^{A_2}(z_2) V_+^{A_3}(z_3) \ldots V_+^{A_n}(z_n) | 0 \rangle, \tag{4.7}$$

where, by (3.21) and (3.22), and because Y^I does not occur in $V^A(z)$, we can replace $Z^I(z)$ by $Z_0^I + zZ_{-1}^I$ (as the other terms in $Z^I(z)$ will annihilate on the vacuum on either the left or the right). Thus, for $d = 1$, by (3.27), the expression (4.7) involves integrals over the 8 bosonic variables $\lambda_0^a, \lambda_{-1}^a, \mu_0^a, \mu_{-1}^a$, $a = 1, 2$, as well as evaluating the dependence on the 8 fermionic variables ψ_0^M, ψ_{-1}^M, $1 \leq M \leq 4$, using (3.23).

The vacuum expectation value of currents

$$\langle 0 | J^{A_1}(z_1) J^{A_2}(z_1) \ldots J^{A_n}(z_n) | 0 \rangle \tag{4.8}$$

can be written as a sum of a number of contributions, one of which has the form

$$\frac{f^{A_1A_2\ldots A_n}}{(z_1 - z_2)(z_2 - z_3)\ldots(z_n - z_1)} \tag{4.9}$$

and we use this contribution in what follows [22].

Thus the $d = 1$ tree amplitude has the form

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{--+\ldots+} = \int \textstyle\prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{ra} \delta({\pi_r}^a - k_r \lambda^a(z)) e^{ik_r \mu^a(z_r)\bar{\pi}_{ra}} \\ \times k_1^4 k_2^4 \langle 0| e^{q_0} \psi^1(z_1)\psi^2(z_1)\psi^3(z_1)\psi^4(z_1)\psi^1(z_2)\psi^2(z_2)\psi^3(z_2)\psi^4(z_2)|0\rangle \\ \times \textstyle\prod_a d^2\lambda^a d^2\mu^a \frac{f^{A_1 A_2 \ldots A_n} dz_1 dz_2 \ldots dz_n}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)} \Big/ d\gamma_M d\gamma_S\end{aligned}$$

Using that, for a component of the fermion field taken alone we would have

$$\langle 0|e^{q_0}\psi^1(z_1)\psi^1(z_2)|0\rangle = (z_2 - z_1)\langle 0|e^{q_0}\psi^1_{-1}\psi^1_0|0\rangle = z_1 - z_2,$$

and integrating over μ^a_0, μ^a_{-1},

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{--+\ldots+} =& \int \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{r=1}^{n}\prod_{a=1}^{2} \delta({\pi_r}^a - k_r\lambda^a(z_r)) \prod_{a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r \bar{\pi}_{ra}\right) \delta\left(\sum_{r=1}^{n} k_r z_r \bar{\pi}_{ra}\right) \prod_{a=1}^{2} d^2\lambda^a \\ &\times k_1^4 k_2^4 (z_1 - z_2)^4 \frac{f^{A_1 A_2 \ldots A_n} dz_1 dz_2 \ldots dz_n}{(z_1 - z_2)(z_2 - z_3)\ldots(z_n - z_1)} \Big/ d\gamma_M d\gamma_S \\ =& \int \prod_{r=1}^{n} \frac{1}{{\pi_r}^1} \prod_{r=1}^{n} \delta\left({\pi_r}^2 - \frac{\lambda^2(z_r)}{\lambda^1(z_r)}{\pi_r}^1\right) \prod_{a=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{{\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)}\right) \delta\left(\sum_{r=1}^{n} \frac{z_r {\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)}\right) \prod_{a=1}^{2} d^2\lambda^a \\ &\times \left(\frac{{\pi_1}^1 {\pi_2}^1 (z_1 - z_2)}{\lambda^1(z_1)\lambda^1(z_2)}\right)^4 \frac{f^{A_1 A_2 \ldots A_n} dz_1 dz_2 \ldots dz_n}{(z_1 - z_2)(z_2 - z_3)\ldots(z_n - z_1)} \Big/ d\gamma_M d\gamma_S\end{aligned}$$

using the delta functions $\delta({\pi_r}^1 - k_r\lambda^1(z_r))$ to perform the k_r integrations again. Noting that when ${\pi_r}^2 - (\lambda^2(z_r)/\lambda^1(z_r)){\pi_r}^1 = 0$, for $b = 1, 2$

$$\sum_{r=1}^{n} {\pi_r}^b \bar{\pi}_{ra} = \sum_{r=1}^{n} \frac{\lambda^b(z_r){\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)} = \lambda^b_0 \sum_{r=1}^{n} \frac{{\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)} + \lambda^b_{-1} \sum_{r=1}^{n} \frac{z_r {\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)},$$

the delta functions

$$\prod_{a=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{{\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)}\right) \delta\left(\sum_{r=1}^{n} \frac{z_r {\pi_r}^1 \bar{\pi}_{ra}}{\lambda^1(z_r)}\right)$$

can be replaced by the momentum conserving delta function, together with a Jacobian factor,

$$(\lambda^1_0 \lambda^2_{-1} - \lambda^1_{-1}\lambda^2_0)^2 \delta^4(\Sigma {\pi_r}^a \bar{\pi}_{rb}).$$

Then, with

$$\zeta_r = \frac{\lambda^2(z_r)}{\lambda^1(z_r)} = \frac{\lambda^2_0 + \lambda^2_{-1} z}{\lambda^1_0 + \lambda^1_{-1} z}, \tag{4.10}$$

using Möbius invariance, we have that

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{--+\ldots+} =& \int \prod_{r=1}^{n} \frac{1}{{\pi_r}^1} \prod_{r=1}^{n} \delta({\pi_r}^2 - \zeta_r {\pi_r}^1) \delta^4(\Sigma {\pi_r}^a \bar{\pi}_{rb}) \frac{d^2\lambda^1 d^2\lambda^2}{(\lambda^1_0\lambda^2_{-1} - \lambda^1_{-1}\lambda^2_0)^2} \\ &\times \left[{\pi_1}^1 {\pi_2}^1 (\zeta_1 - \zeta_2)\right]^4 \frac{f^{A_1 A_2 \ldots A_n} d\zeta_1 d\zeta_2 \ldots d\zeta_n}{(\zeta_1 - \zeta_2)(\zeta_2 - \zeta_3)\ldots(\zeta_n - \zeta_1)} \Big/ d\gamma_M d\gamma_S\end{aligned} \tag{4.11}$$

Noting that we can write

$$\frac{d^2\lambda^1 d^2\lambda^2}{(\lambda_0^1\lambda_{-1}^2 - \lambda_{-1}^1\lambda_0^2)^2} = d\gamma_M d\gamma_S,$$

because the left hand side is the invariant measure on the product of the Möbius and scaling groups, and using the $\delta({\pi_r}^2 - \zeta_r {\pi_r}^1)$ to do the ζ_r integrations,

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{--+\ldots+} &= \delta^4({\pi_r}^a \bar{\pi}_{rb}) \left({\pi_1}^2{\pi_2}^1 - {\pi_2}^2{\pi_1}^1\right)^4 f^{A_1A_2\ldots A_n}\prod_{r=1}^{n}\left({\pi_r}^2{\pi_{r+1}}^1 - {\pi_{r+1}}^2{\pi_r}^1\right)^{-1} \\ &= \delta^4({\pi_r}^a\bar{\pi}_{rb})\frac{\langle 1,2\rangle^4 f^{A_1A_2\ldots A_n}}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle},\end{aligned} \tag{4.12}$$

with $\pi^a_{n+1} \equiv \pi^a_1$.

Of course, this result could also be obtained within the path integral framework. From this point of view, after integrating out the $Y^{I\mu}$ fields, the $d=1$ sector path integral is

$$\begin{aligned}A^{\text{tree}} = \sum_{d=1}\int & DZ_I\ \delta((\partial_{\bar{z}} - iA_{\bar{z}})Z^I) \\ &\cdot \int\prod_{i=1}^{n} dz_i \int D\phi_G e^{-S_G} J^{A_1}(z_1)J^{A_2}(z_2)\ldots J^{A_n}(z_n) \\ &\cdot \prod_{r=1}^{n}\frac{dk_r}{k_r}\prod_{r,a}\delta(\pi_r^a - k_r\lambda^a(z_r))\ e^{ik_r\mu^a(z_r)\bar{\pi}_{ra}} \\ &\cdot \left[A_{+r} + k_r^4\psi^1(z_r)\psi^2(z_r)\psi^3(z_r)\psi^4(z_r)A_{-r}\right]\Big/ d\gamma_M d\gamma_S.\end{aligned} \tag{4.13}$$

As discussed in section 2, we work in a gauge where the gauge potentials are zero, so the path satisfies $\partial_{\bar{z}}Z^I = 0$, and $Z^I(z) = Z_0^I + Z_1^I z$ from (2.15).

Computing the path integral by replacing $DZ_I\ \delta((\partial_{\bar{z}} - iA_{\bar{z}}^{(d=1)})Z^I)$ with $\prod_{I=1}^{8} dZ_0^I dZ_1^I$, and performing the integrals over Z_0^I, Z_1^I, where the fields $\lambda^a(z), \mu^a(z), \psi^M(z)$ are now given by the solutions $Z^I(z) = Z_0^I + zZ_1^I$, we see that evaluating the path integral gives the result obtained from the canonical quantization. Identifying the integration variables Z_0^I, Z_1^I with the variables $\lambda^a_{0,-1}, \mu^a_{0,-1}, \psi^M_{0,-1}$, and performing the Grassmann integration $\int d\psi_0\psi_0 = 1, \int d\psi_{-1}\psi_{-1} = 1$, for each M, we find that for the two negative helicities in the positions $1,2$, we obtain (4.12), together with a factor of $A_{-1}A_{-2}A_{+3}\ldots A_{+n}$ corresponding to the polarizations in (4.13).

5. Loop amplitudes

The n-point loop amplitude is a sum over contributions corresponding to instanton numbers d. We write the integrand of such a contribution as a product of factors:

$$\mathcal{A}^{\text{loop}}_{n,d} = \int \mathcal{A}^{\lambda\mu}_{n,d}\mathcal{A}^{\psi}_{n,d}\mathcal{A}^{J^A}_{n}\mathcal{A}^{\text{ghost}}\frac{d\alpha_0\, d\tau}{2\pi \text{Im}\tau}\prod_{r=1}^{n}\rho_r d\nu_r, \qquad \rho_r = e^{2\pi i\nu_r}, \quad w = e^{2\pi i\tau}, \tag{5.1}$$

where $\mathcal{A}^{\lambda\mu}_{n,d}$ is the part of the integrand associated with the bosonic twistor fields λ, μ; $\mathcal{A}^{\psi}_{n,d}$ is the part associated with the fermionic twistor fields ψ; $\mathcal{A}^{J^A}_{n}$ is the part associated with the current algebra J^A; $\mathcal{A}^{\rm ghost}$ is the part associated with the ghost fields; and τ is pure imaginary. The integration with respect to α_0 is the averaging over the gauge transformation u^{a_0} referred to in (3.18); the normalization factor $2\pi{\rm Im}\tau$ will be explained below. We shall restrict our attention to MHV amplitudes. For these, the fermionic part $\mathcal{A}^{\psi}_{n,d}$ will vanish unless $d=2$ and so we shall use this value below.

Twistor bosonic contribution to the loop integrand. Using the vertices (3.35) and (3.36), the part of the integrand for the n-particle loop amplitude associated with the bosonic twistor fields, λ, μ, is

$$\begin{aligned}\mathcal{A}^{\lambda\mu}_{n,2} = &\int {\rm tr}\left(e^{2q_0}u^{a_0}\prod_{r=1}^{n}\exp\left\{ik_r\lambda^a(\rho_r)\bar{\omega}_{ra}+ik_r\mu^a(\rho_r)\bar{\pi}_{ra}\right\}w^{L_0}\right)\\ &\times\prod_{r=1}^{n}\frac{dk_r}{k_r}\prod_{a=1}^{2}e^{-i\bar{\omega}_{ra}\pi_r{}^a}d\bar{\omega}_{ra}/d\gamma_S,\end{aligned} \tag{5.2}$$

Here we have rewritten the delta functions $\delta(k_r\lambda^a(\rho_r)-\pi_r{}^a)$ as Fourier transforms on $\bar{\omega}_{ra}$ in order that we can perform the trace on the bosonic variables λ^a, using the relation (C.1) from appendix C

$${\rm tr}\left(e^{dq_0}u^{a_0}\prod_{j=1}^{n}e^{i\omega_j Z(\rho_j)}w^{L_0}\right) = u^{(d+1)/2}\prod_{i=1}^{d}\delta\left(\sum_{j=1}^{n}F_i^d(\hat{\rho}_j,w)\omega_j\right), \tag{5.3}$$

where $\hat{\rho}_j = u^{-\frac{1}{2}}\rho_j = e^{2\pi i\hat{\nu}_j}$, $\hat{\nu}_j=\nu_j+i\alpha_0/4\pi$, and

$$\begin{aligned}F_k^d(\rho,w) &= \sum_{n=-\infty}^{\infty} w^{\frac{1}{2}(n-1)(d(n-2)+2k)}\left(\frac{\rho}{w}\right)^{d(1-n)-k}\\ &= \rho^{d/2}w^{d/8-k/4}\,\theta\begin{bmatrix}2k/d-1\\0\end{bmatrix}(-d\nu,d\tau),\end{aligned} \tag{5.4}$$

using the notation of (2.22), so that

$$F_1^2(\rho,w)=\rho\theta_3(2\nu,2\tau),\qquad F_2^2(\rho,w)=w^{-\frac{1}{4}}\rho\theta_2(2\nu,2\tau), \tag{5.5}$$

From this we have

$$\begin{aligned}\mathcal{A}^{\lambda\mu}_{n,2} = u^6 &\int\prod_{i,a=1}^{2}\delta\left(\sum_{r=1}^{n}k_rF_i^2(\hat{\rho}_r,w)\bar{\omega}_{ra}\right)\delta\left(\sum_{r=1}^{n}k_rF_i^2(\hat{\rho}_r,w)\bar{\pi}_{ra}\right)\\ &\times\prod_{r=1}^{n}\frac{dk_r}{k_r}\prod_{a=1}^{2}e^{-i\bar{\omega}_{ra}\pi_r{}^a}d\bar{\omega}_{ra}/d\gamma_S.\end{aligned} \tag{5.6}$$

for, putting $d=2$ in (5.3), we see that we get a factor of $u^{\frac{3}{2}}$ for each of the four bosonic components of Z. Expressing the second delta functions as Fourier transforms on $\tilde{\lambda}_i^a$,

$$\mathcal{A}^{\lambda\mu}_{n,2} = u^6\int\prod_{i,a=1}^{2}\delta\left(\sum_r k_rF_i^2(\hat{\rho}_r,w)\bar{\pi}_{ra}\right)$$

$$\times \exp\left(i \sum_{r=1}^{n} \sum_{a=1}^{2} \left[\sum_{i=1}^{2} k_r \tilde{\lambda}_i^a F_i^2(\hat{\rho}_r, w) \bar{\omega}_{ra} - i \bar{\omega}_{ra} {\pi_r}^a \right] \right) \prod_{a=1}^{2} d^2 \tilde{\lambda}^a \prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{a=1}^{2} d\bar{\omega}_{ra} / d\gamma_S$$

$$= u^6 \int \prod_{r=1}^{n} \prod_{a=1}^{2} \delta\Big(k_r \tilde{\lambda}^a(\hat{\rho}_r, w) - {\pi_r}^a \Big) \prod_{i,a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r F_i^2(\hat{\rho}_r, w) \bar{\pi}_{ra} \right) \prod_{a=1}^{2} d^2 \tilde{\lambda}^a \prod_{r=1}^{n} \frac{dk_r}{k_r} / d\gamma_S,$$

where we have performed the integrals over $\bar{\omega}_{ra}$ and set

$$\tilde{\lambda}^a(\hat{\rho}_r, w) = \tilde{\lambda}_1^a F_1^2(\hat{\rho}_r, w) + \tilde{\lambda}_2^a F_2^2(\hat{\rho}_r, w). \tag{5.7}$$

Performing the k_r integrations using the $\delta\left(k_r \tilde{\lambda}^1(\hat{\rho}_r, w) - {\pi_r}^1 \right)$ delta functions,

$$\mathcal{A}_{n,2}^{\lambda\mu} = u^6 \int \prod_{r=1}^{n} \frac{1}{{\pi_r}^1} \delta\left(\frac{\tilde{\lambda}^2(\hat{\rho}_r, w)}{\tilde{\lambda}^1(\hat{\rho}_r, w)} {\pi_r}^1 - {\pi_r}^2 \right) \prod_{i,a=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{F_i^2(\hat{\rho}_r, w)}{\tilde{\lambda}^1(\hat{\rho}_r, w)} {\pi_r}^1 \bar{\pi}_{ra} \right) \prod_{a=1}^{2} d^2 \tilde{\lambda}^a / d\gamma_S.$$

As before, given the constraints $\tilde{\lambda}^2(\hat{\rho}_r, w) {\pi_r}^1 = \tilde{\lambda}^1(\hat{\rho}_r, w) {\pi_r}^2$, we can replace

$$\prod_{i,a=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{F_i^2(\hat{\rho}_r, w)}{\tilde{\lambda}^1(\hat{\rho}_r, w)} {\pi_r}^1 \bar{\pi}_{ra} \right) \quad \text{by} \quad (\tilde{\lambda}_1^1 \tilde{\lambda}_2^2 - \tilde{\lambda}_2^1 \tilde{\lambda}_1^2)^2 \delta^4(\Sigma \pi_r \bar{\pi}_r),$$

so that

$$\mathcal{A}_{n,2}^{\lambda\mu} = \delta^4(\Sigma \pi_r \bar{\pi}_r) u^6 \int (\tilde{\lambda}_1^1 \tilde{\lambda}_2^2 - \tilde{\lambda}_2^1 \tilde{\lambda}_1^2)^2 \prod_{r=1}^{n} \frac{1}{{\pi_r}^1} \delta\left(\tilde{\xi}(\hat{\nu}_r, \tau) {\pi_r}^1 - {\pi_r}^2 \right) \prod_{a=1}^{2} d^2 \tilde{\lambda}^a / d\gamma_S, \tag{5.8}$$

where

$$\begin{aligned} \tilde{\xi}(\hat{\nu}, \tau) &= \frac{\tilde{\lambda}^2(\hat{\rho}, w)}{\tilde{\lambda}^1(\hat{\rho}, w)} = \frac{\tilde{\lambda}_1^2 F_1^2(\hat{\rho}, w) + \tilde{\lambda}_2^2 F_2^2(\hat{\rho}, w)}{\tilde{\lambda}_1^1 F_1^2(\hat{\rho}, w) + \tilde{\lambda}_2^1 F_2^2(\hat{\rho}, w)} \\ &= \frac{\tilde{\lambda}_1^2 \xi(\hat{\nu}, \tau) + \tilde{\lambda}_2^2}{\tilde{\lambda}_1^1 \xi(\hat{\nu}, \tau) + \tilde{\lambda}_2^1}, \qquad \text{for } \xi(\hat{\nu}, \tau) = \frac{F_1^2(\hat{\rho}, w)}{F_2^2(\hat{\rho}, w)}. \end{aligned} \tag{5.9}$$

Now we use the delta functions $\delta\left(\tilde{\xi}(\hat{\nu}_r, \tau) {\pi_r}^1 - {\pi_r}^2 \right)$ to do the integrations over ν_r in (5.1):

$$\begin{aligned} \mathcal{A}_{n,2}^{\text{loop}} &= \delta^4(\Sigma \pi_r \bar{\pi}_r) \int u^6 \prod_{r=1}^{n} \frac{1}{({\pi_r}^1)^2 \tilde{\xi}'(\hat{\nu}_r, \tau)} \\ &\quad \times (\tilde{\lambda}_1^1 \tilde{\lambda}_2^2 - \tilde{\lambda}_2^1 \tilde{\lambda}_1^2)^2 \rho_\Pi \mathcal{A}_{n,d}^{\psi} \mathcal{A}_n^{J^A} \mathcal{A}^{\text{ghost}} \frac{d\alpha_0 \, d\tau}{2\pi \text{Im}\tau} \prod_{a=1}^{2} d^2 \tilde{\lambda}^a / d\gamma_S, \end{aligned} \tag{5.10}$$

where $\rho_\Pi = \prod_{r=1}^{n} \rho_r$ and

$$\tilde{\xi}'(\hat{\nu}, \tau) \equiv \frac{\partial \tilde{\xi}(\hat{\nu}, \tau)}{\partial \hat{\nu}}.$$

It is to be understood that in the rest of the integrand k_r and ν_r are determined by

$$k_r = \frac{{\pi_r}^1}{\tilde{\lambda}^1(\hat{\rho}_r, w)}, \qquad \tilde{\xi}(\hat{\nu}_r, \tau) = \frac{{\pi_r}^2}{{\pi_r}^1}, \tag{5.11}$$

so the second equation determines ν_r as a function of $\tilde{\lambda}_i^a, \tau, \alpha_0$ and π_r^2/π_r^1.

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Twistor fermionic contribution to the loop integrand. Now consider the fermionic part of the integrand,

$$\mathcal{A}^{\psi}_{n,2} = k_1^4 k_2^4 \mathrm{tr}\big(e^{2q_0}u^{a_0}(-1)^{a_0}\psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1)\psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2)w^{L_0}\big)\,. \tag{5.12}$$

If we consider one component of the fermion field $\psi^M(\rho)$ taken in isolation,

$$\begin{aligned}
\mathrm{tr}\left(e^{2q_0}u^{a_0}(-1)^{a_0}\psi^1(\rho_1)\psi^1(\rho_2)w^{L_0}\right) &= u^{-\frac{3}{2}}F_1^2(\rho_1,w)F_2^2(\rho_2,w) - F_2^2(\rho_1,w)F_1^2(\rho_2,w)\\
&= -u^{-\frac{3}{2}}(\tilde{\xi}(\nu_1',\tau) - \tilde{\xi}(\nu_2',\tau))\tfrac{\tilde{\lambda}^1(\rho_1,w)\tilde{\lambda}^1(\rho_2,w)}{\tilde{\lambda}_1^1\tilde{\lambda}_2^2-\tilde{\lambda}_2^1\tilde{\lambda}_1^2}\\
&= -\tfrac{u^{-\frac{3}{2}}\langle 1,2\rangle}{k_1k_2(\tilde{\lambda}_1^1\tilde{\lambda}_2^2-\tilde{\lambda}_2^1\tilde{\lambda}_1^2)}
\end{aligned}$$

so that

$$\mathcal{A}^{\psi}_{n,2} = \frac{u^{-6}\langle 1,2\rangle^4}{(\tilde{\lambda}_1^1\tilde{\lambda}_2^2 - \tilde{\lambda}_2^1\tilde{\lambda}_1^2)^4}. \tag{5.13}$$

Thus, from (5.10), we see that the factors of u cancel and

$$\mathcal{A}^{\mathrm{loop}}_{n,2} = \langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)\int\left[\prod_{r=1}^{n}\frac{1}{(\pi_r{}^1)^2\,\tilde{\xi}'(\hat{\nu}_r,\tau)}\right]\frac{\mathcal{A}_n^{J^A}\mathcal{A}^{\mathrm{ghost}}}{(\tilde{\lambda}_1^1\tilde{\lambda}_2^2 - \tilde{\lambda}_2^1\tilde{\lambda}_1^2)^2}\rho_{\Pi}\frac{d\alpha_0\,d\tau}{2\pi\mathrm{Im}\tau}\prod_{a=1}^{2}d^2\tilde{\lambda}^a/d\gamma_S. \tag{5.14}$$

Since

$$\begin{aligned}
\tilde{\xi}(\nu,\tau) &= \frac{\tilde{\lambda}_1^2\xi(\nu,\tau) + \tilde{\lambda}_2^2}{\tilde{\lambda}_1^1\xi(\nu,\tau) + \tilde{\lambda}_2^1},\\
\prod_{r=1}^{n}\frac{\xi'(\hat{\nu}_r,\tau)}{\xi(\hat{\nu}_r,\tau) - \xi(\hat{\nu}_{r+1},\tau)} &= \prod_{r=1}^{n}\frac{\tilde{\xi}'(\hat{\nu}_r,\tau)}{\tilde{\xi}(\hat{\nu}_r,\tau) - \tilde{\xi}(\hat{\nu}_{r+1},\tau)}\\
&= \prod_{r=1}^{n}\frac{(\pi_r{}^1)^2\tilde{\xi}'(\hat{\nu}_r,\tau)}{\pi_r{}^2\pi_{r+1}{}^1 - \pi_{r+1}{}^2\pi_r{}^1} = \prod_{r=1}^{n}\frac{(\pi_r{}^1)^2\tilde{\xi}'(\hat{\nu}_r,\tau)}{\langle r,r+1\rangle}
\end{aligned} \tag{5.15}$$

Using this in (5.14),

$$\mathcal{A}^{\mathrm{loop}}_{n,2} = \frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle}\times\int\left[\prod_{r=1}^{n}\frac{\xi(\hat{\nu}_r,\tau)-\xi(\hat{\nu}_{r+1},\tau)}{\xi'(\hat{\nu}_r,\tau)}\right]\frac{\mathcal{A}_n^{J^A}\mathcal{A}^{\mathrm{ghost}}}{(\tilde{\lambda}_1^1\tilde{\lambda}_2^2-\tilde{\lambda}_2^1\tilde{\lambda}_1^2)^2}\prod_{a=1}^{2}d^2\tilde{\lambda}^a\frac{\rho_{\Pi}}{d\gamma_S}\frac{d\alpha_0 d\tau}{2\pi\mathrm{Im}\tau}. \tag{5.16}$$

which separates out from the rest of the amplitude the kinematic factor present in the tree amplitude.

From (5.5) we have that

$$\xi(\nu,\tau) = w^{\frac{1}{4}}\frac{\theta_3(2\nu,2\tau)}{\theta_2(2\nu,2\tau)}.$$

Using the relations,

$$\begin{aligned}
2\theta_3(2\nu,2\tau)\theta_3(0,2\tau) &= \theta_3(\nu,\tau)^2 + \theta_4(\nu,\tau)^2,\\
2\theta_2(2\nu,2\tau)\theta_2(0,2\tau) &= \theta_3(\nu,\tau)^2 - \theta_4(\nu,\tau)^2,
\end{aligned}$$

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we see that $\xi(\nu,\tau)$ is related by a bilinear transformation, whose coefficients are functions of τ, to $\theta_3(\nu,\tau)^2/\theta_4(\nu,\tau)^2$. Further, using the relation

$$\theta_1(\nu,\tau)^2\theta_2(0,\tau)^2 = \theta_4(\nu,\tau)^2\theta_3(0,\tau)^2 - \theta_3(\nu,\tau)^2\theta_4(0,\tau)^2,$$

we see that $\xi(\nu,\tau)$ is also related by a bilinear transformation to the Weierstrass $\mathcal{P}$ function

$$\mathcal{P}(\nu,\tau) = \frac{\pi^2}{3}\left[\theta_2(0,\tau)^4 - \theta_4(0,\tau)^4\right] + \left[\frac{\theta_1'(0,\tau)\theta_3(\nu,\tau)}{\theta_3(0,\tau)\theta_1(\nu,\tau)}\right]^2. \tag{5.17}$$

Using the bilinear invariance of the integrand, this implies

$$\mathcal{A}^{\mathrm{loop}}_{n,2} = \frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle} \times\int\left[\prod_{r=1}^{n}\frac{\mathcal{P}(\hat{\nu}_r,\tau)-\mathcal{P}(\hat{\nu}_{r+1},\tau)}{\mathcal{P}'(\hat{\nu}_r,\tau)}\right]\frac{\mathcal{A}^{J^A}_n\mathcal{A}^{\mathrm{ghost}}}{(\tilde\lambda^1_1\tilde\lambda^2_2-\tilde\lambda^1_2\tilde\lambda^2_1)^2}\prod_{a=1}^{2}d^2\tilde\lambda^a\frac{\rho_\Pi}{d\gamma_S}\frac{d\alpha_0 d\tau}{2\pi\mathrm{Im}\tau}, \tag{5.18}$$

as an alternative symmetric expression.

Regarding

$$\begin{pmatrix}\tilde\lambda^2_1 & \tilde\lambda^2_2\\ \tilde\lambda^1_1 & \tilde\lambda^1_2\end{pmatrix}$$

as the matrix defining the bilinear transformation that takes $\xi_r = \xi(\hat\nu_r,\tau)$ to ${\pi_r}^2/{\pi_r}^1$ for $1\le r\le 3$, the invariant measure

$$\frac{d^2\tilde\lambda^a}{(\tilde\lambda^1_1\tilde\lambda^2_2-\tilde\lambda^1_2\tilde\lambda^2_1)^2} = \frac{d\xi_1 d\xi_2 d\xi_3}{(\xi_1-\xi_2)(\xi_2-\xi_3)(\xi_3-\xi_1)}d\gamma_S,$$

$$\mathcal{A}^{\mathrm{loop}}_{n,2} = \frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle n,1\rangle}\int\frac{(\xi_3-\xi_4)}{(\xi_3-\xi_1)}\left[\prod_{r=4}^{n}\frac{(\xi_r-\xi_{r+1})}{\xi_r'}\right]\mathcal{A}^{J^A}_n\mathcal{A}^{\mathrm{ghost}}\rho_\Pi d\nu_1 d\nu_2 d\nu_3\frac{d\alpha_0 d\tau}{2\pi\mathrm{Im}\tau}. \tag{5.19}$$

For the first non vanishing amplitude, $n=4$, noting that

$$\frac{\langle 1,2\rangle\langle 3,4\rangle}{\langle 1,4\rangle\langle 3,2\rangle} = -\frac{s}{t}$$

and

$$\int\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)}\delta\left(\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)}+\frac{s}{t}\right)d\nu_4 = \frac{(\xi_3-\xi_4)(\xi_4-\xi_1)}{(\xi_3-\xi_1)\xi_4'},$$

$$\mathcal{A}^{\mathrm{loop}}_{4,2} = -\frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,1\rangle}\frac{s}{t}\int\delta\left(\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)}+\frac{s}{t}\right)\mathcal{A}^{J^A}_4\mathcal{A}^{\mathrm{ghost}}\prod_{r=1}^{4}\rho_r d\nu_r\frac{d\alpha_0 d\tau}{2\pi\mathrm{Im}\tau}. \tag{5.20}$$

Similarly, we can derive a corresponding expression for the n-point loop,

$$\mathcal{A}^{\mathrm{loop}}_{n,2} = \frac{\langle 1,2\rangle^n\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 2,3\rangle^{n-2}\langle 3,1\rangle^{n-4}} \tag{5.21}$$

$$\times\int\prod_{r=4}^{n}\frac{1}{\langle r,1\rangle^2}\delta\left(\frac{(\xi_r-\xi_3)(\xi_2-\xi_1)}{(\xi_r-\xi_1)(\xi_2-\xi_3)}-\frac{\langle r,3\rangle\langle 2,1\rangle}{\langle r,1\rangle\langle 2,3\rangle}\right)\mathcal{A}^{J^A}_n\mathcal{A}^{\mathrm{ghost}}\prod_{r=1}^{n}\rho_r d\nu_r\frac{d\alpha_0 d\tau}{2\pi\mathrm{Im}\tau}.$$

Because $\hat{\nu}_r = \nu_r + i\alpha_0/4\pi$ and the integral is invariant as $\nu_r \to \nu_r + \frac{1}{2}\tau$, we see that it is periodic under $\alpha_0 \to \alpha_0 + 2\pi\mathrm{Im}\tau$. So to average over α_0, we integrate over the range $0 \le \alpha_0 \le 2\pi\mathrm{Im}\tau$ and divide by $2\pi\mathrm{Im}\tau$, which explains the normalization factor introduced in (5.1).

Path integral derivation. In the path integral approach, working in the gauge $A_i = 0$, we use the paths given in (2.24) with $n = 2$ and $\lambda^a(\rho) = Z^a(\nu), \mu^a(\rho) = Z^{a+2}(\nu)$, $a = 1, 2; \psi^M(\rho) = Z^{M+4}(\nu), 1 \le M \le 4$.

The path integral on the cylinder includes, up to normalization,

$$\int \mathcal{A}^{\lambda\mu}_{n,2} \prod_{r=1}^{n} d\nu_r = \int \prod_{I=1}^{4} dc_0^I dc_1^I \sum_{\epsilon'=0}^{1} \int_0^2 d\epsilon \left(\prod_{r=1}^{n} \exp\left\{ ik_r \lambda^a(\rho_r)\bar{\omega}_{ra} + ik_r \mu^a(\rho_r)\bar{\pi}_{ra} \right\} \right)$$
$$\times \left(\prod_{r=1}^{n} \frac{dk_r}{k_r} \prod_{a=1}^{2} e^{-i\bar{\omega}_{ra}\pi_r{}^a} d\bar{\omega}_{ra} (\prod_{r=1}^{n-1} d\nu_r) \right) \Big/ d\gamma_S. \tag{5.22}$$

Performing the integrations over $c_0^I, c_1^I, 1 \le I \le 4$, we find a formula analogous to the canonical expression (5.6), save that the functions $F_1^2(\rho_r, w)$ and $F_2^2(\rho_r, w)$ are replaced with $\theta[{\frac{1}{2}\epsilon \atop \epsilon'}](2\nu_r, 2\tau)$, and $\theta[{\frac{1}{2}\epsilon+1 \atop \epsilon'}](2\nu_r, 2\tau)$, respectively, and the α_0 integration is exchanged for a sum over ϵ' and the ϵ integration.

The calculation proceeds as before to a formula similar to (5.8), except that $\tilde{\xi}(\nu, \tau)$ is replaced with

$$\tilde{\eta}(\nu, \tau) = \frac{\tilde{\lambda}_1^2 \eta(\nu, \tau) + \tilde{\lambda}_2^2}{\tilde{\lambda}_1^1 \eta(\nu, \tau) + \tilde{\lambda}_2^1} \quad \text{where} \quad \eta(\nu, \tau) = \frac{\theta[{\frac{1}{2}\epsilon \atop \epsilon'}](2\nu, 2\tau)}{\theta[{\frac{1}{2}\epsilon+1 \atop \epsilon'}](2\nu, 2\tau)} = \frac{\theta_3(2\nu + \frac{1}{2}\epsilon' + \frac{1}{2}\epsilon\tau, 2\tau)}{\theta_2(2\nu + \frac{1}{2}\epsilon' + \frac{1}{2}\epsilon\tau, 2\tau)}.$$

After integrating with respect to ϵ, the resulting integrand depends on the differences $\nu_i - \nu_j$, and is independent of ϵ'. Then $\tilde{\eta}(\nu, \tau)$ is essentially $\tilde{\xi}(\nu, \tau)$ from (5.9), since the factor $w^{\frac{1}{4}}$ in $\xi(\nu, \tau)$ can be absorbed in the $\tilde{\lambda}^a$ integrations, and $\epsilon\tau/2$ can be replaced by $i\alpha_0/2\pi$, $d\epsilon = d\alpha_0/\pi\mathrm{Im}\tau$ (α_0, ϵ real, τ pure imaginary).

In analogy with (5.10), we use the delta functions $\delta\left(\tilde{\eta}(\nu_r, \tau)\pi_r{}^1 - \pi_r{}^2\right)$ to do the integrations over ν_r:

$$\int \mathcal{A}^{\lambda\mu}_{n,2} \ \prod_{r=1}^{n} d\nu_r$$
$$= \delta^4(\Sigma \pi_r \bar{\pi}_r)\, 2 \int_0^2 d\epsilon \int \left[\prod_{r=1}^{n} \frac{1}{(\pi_r{}^1)^2 \tilde{\eta}'(\nu_r, \tau)} \right] (\tilde{\lambda}_1^1 \tilde{\lambda}_2^2 - \tilde{\lambda}_2^1 \tilde{\lambda}_1^2)^2 \left[\prod_{a=1}^{2} d^2 \tilde{\lambda}^a \right] \rho_\Pi \Big/ d\gamma_S.$$

In the rest of the integrand k_r and ν_r are determined by $k_r = \pi_r{}^1/\tilde{\lambda}^1(\rho_r, w)$ and $\tilde{\eta}(\nu_r, \tau) = \pi_r{}^2/\pi_r{}^1$. The one-loop integrand for the fermionic fields, for one permutation of helicities, is

$$\mathcal{A}^{\psi}_{n,2} = k_1^4 k_2^4 \int \prod_{M=1}^{4} dc_0^{M+4} dc_1^{M+4} \psi^M(\rho_1) \psi^M(\rho_2) = \frac{\langle 1, 2 \rangle^4}{(\tilde{\lambda}_1^1 \tilde{\lambda}_2^2 - \tilde{\lambda}_2^1 \tilde{\lambda}_1^2)^4}.$$

Thus we obtain (5.16) as before, apart from a factor of 4.

Ghost contribution to the loop integrand. The ghost fields in the theory are the customary ghost fields, b, c, with conformal spin $(2, -1)$, associated with the reparametrization invariance, and the ghosts for the $\mathrm{U}(1)$ gauge fields u, v with conformal spin $(1, 0)$.

The partition function for a general fermionic "b, c" system with conformal dimensions λ and $1 - \lambda$ respectively is

$$\mathrm{tr}(b_0 c_0 \omega^{L_0 - \frac{c}{24}}(-1)^F) = \omega^{-\frac{c}{24}} \omega^{\frac{1}{2}\lambda(1-\lambda)} \prod_{n=1}^{\infty} (1 - \omega^n)^2 \,,$$

where the central charge is $12\lambda(1-\lambda) - 2$, and $L(z) = -\lambda \,{}^{\times}_{\times} b(z) c'(z) {}^{\times}_{\times} + (1-\lambda) \,{}^{\times}_{\times} b'(z) c(z) {}^{\times}_{\times}$. So

$$\mathrm{tr}(b_0 c_0 \omega^{L_0 - \frac{c}{24}}(-1)^F) = \omega^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - \omega^n)^2 = \eta(\tau)^2 \,.$$

and the reparametrization and $\mathrm{U}(1)$ ghosts each contribute this factor to the integrand:

The factor of $(-1)^F$ is included so that the ghosts have the same periodicity as the original coordinate transformations. See Freeman and Olive [23]. The b_0, c_0 insertion projects onto half the states in the b, c system; without this projection, the $(-1)^F$ would force the trace to vanish.

So the total ghost contribution is

$$\mathcal{A}^{\text{ghost}} = \eta(\tau)^4. \tag{5.23}$$

6. Current algebra loop

For the final piece of the integrand of the loop amplitude, we compute the one-loop amplitude,

$$\mathrm{tr}(J^{a_1}(\rho_1) J^{a_2}(\rho_2) \ldots J^{a_n}(\rho_n) w^{L_0}), \tag{6.1}$$

for the current algebra of an arbitrary Lie group, G,

$$[J^a_m, J^b_n] = i f^{abc} J^c_{m+n} + km\delta^{ab}\delta_{m,-n}, \qquad J^a(\rho) = \sum_n J^a_n \rho^{-n-1}. \tag{6.2}$$

We calculate this using a recursion relation, and will present the derivation in a future publication [19]. Relevant discussion is also given in [24, 25]. The amplitudes can be expressed in terms of the τ-dependent invariant tensors,

$$\mathrm{tr}(J^{a_1}_0 J^{a_2}_0 \ldots J^{a_n}_0 w^{L_0}), \tag{6.3}$$

which themselves can be expressed as products of (constant) invariant tensors and functions of τ. We note that (6.1) is symmetric under simultaneous identical permutations of the ρ_j and a_j. The structure of the loop amplitude for $n \geq 4$ is best understood by considering first the zero, two and three-point functions. (The one-point function necessarily vanishes.) In what follows L_0 denotes the zero-mode generator for the Virasoro algebra associated with the current algebra.

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Fermionic representations. We begin by considering the case where $J^a(\rho)$ is given by a fermionic representation,

$$J^a(\rho) = \frac{i}{2} T^a_{ij} b^i(\rho) b^j(\rho) \qquad J^a_n = \frac{i}{2} \sum_r T^a_{ij} b^i_r b^j_{n-r} \tag{6.4}$$

$$[T^a, T^b] = f^{abc} T^c, \qquad \mathrm{tr}(T^a T^b) = -2k\delta^{ab}, \qquad b^j(\rho) = \sum_r b^j_r \rho^{-r-\frac{1}{2}}, \tag{6.5}$$

where T^a, $1 \le a \le \dim G$, are real antisymmetric matrices providing a real dimension D representation of G, and the $b^i_r, r \in \mathbb{Z} + \frac{1}{2}$, are Neveu-Schwarz fermionic oscillators, $\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r,-s}$, $\quad b^j_r|0\rangle = 0$, $r > 0$, $\quad (b^j_r)^\dagger = b^j_{-r}$.

For this representation, the zero-point function,

$$\chi(\tau) = \mathrm{tr}\left(w^{L_0}\right) = \prod_{s=\frac{1}{2}}^{\infty} (1 + w^s)^D, \tag{6.6}$$

while the one-point function vanishes, $\mathrm{tr}(J^a(\rho) w^{L_0})$, as it does for any representation.

The n-point one-loop current algebra amplitude is computed by using the usual recurrence relation, for calculating $\mathrm{tr}(b^{i_1}_{r_1} \dots b^{i_n}_{r_n} w^{L_0})$ in the free fermion representation, obtained by moving $b^{i_n}_{r_n}$ around the trace,

$$\mathrm{tr}(b^{i_1}_{r_1} \dots b^{i_n}_{r_n} w^{L_0}) = \frac{w^{r_n}}{1 + w^{r_n}} \sum_{m=1}^{n-1} (-1)^{m+1} \delta_{r_m, -r_n} \delta^{i_m i_n} \mathrm{tr}(b^{i_1}_{r_1} \dots b^{i_{m-1}}_{r_{m-1}} b^{i_{m+1}}_{r_{m+1}} \dots b^{i_{n-1}}_{r_{n-1}} w^{L_0}).$$

Using this, we obtain the two-point loop,

$$\mathrm{tr}(J^a(\rho_1) J^b(\rho_2) w^{L_0}) = \frac{k\chi(\tau)}{\rho_1\rho_2} \delta^{ab} \chi_F(\nu_1 - \nu_2, \tau)^2, \tag{6.7}$$

where

$$\chi_F(\nu, \tau) = \sum_{r=\frac{1}{2}}^{\infty} \frac{e^{2\pi i r \nu} + w^r e^{-2\pi i r \nu}}{1 + w^r} = \frac{i}{2} \theta_2(0,\tau)\theta_4(0,\tau) \frac{\theta_3(\nu,\tau)}{\theta_1(\nu,\tau)}. \tag{6.8}$$

The three-point loop,

$$\mathrm{tr}(J^a(\rho_1) J^b(\rho_2) J^c(\rho_3) w^{L_0}) = \frac{ik f^{abc} \chi(\tau)}{\rho_1\rho_2\rho_3} \chi_F^{21} \chi_F^{32} \chi_F^{13}, \tag{6.9}$$

and the four-point loop,

$$\begin{aligned}
&\mathrm{tr}(J^a(\rho_1) J^b(\rho_2) J^c(\rho_3) J^d(\rho_4) w^{L_0}) \\
&= \frac{\chi(\tau)}{\rho_1\rho_2\rho_3\rho_4} [\sigma^{abcd} \chi_F^{12}\, \chi_F^{23}\, \chi_F^{34}\, \chi_F^{14} - \sigma^{abdc} \chi_F^{12}\, \chi_F^{24}\, \chi_F^{34}\, \chi_F^{13} - \sigma^{acbd} \chi_F^{13}\, \chi_F^{23}\, \chi_F^{24}\, \chi_F^{14} \\
&\quad + k^2\, \delta^{ab} \delta^{cd}\, (\chi_F^{12})^2\, (\chi_F^{34})^2 + k^2\, \delta^{ac} \delta^{bd}\, (\chi_F^{13})^2\, (\chi_F^{24})^2 + k^2 \delta^{ad}\, \delta^{bc}\, (\chi_F^{14})^2\, (\chi_F^{23})^2].
\end{aligned} \tag{6.10}$$

where $\chi_F^{ij} = \chi_F(\nu_j - \nu_i, \tau)$ and

$$\sigma^{abcd} = \mathrm{tr}(T^a T^b T^c T^d). \tag{6.11}$$

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General representations. The amplitude (6.1) can be evaluated by means of recurrence relations between functions

$$\mathrm{tr}\left(J_0^{a_1}J_0^{a_2}\ldots J_0^{a_m}J^{b_1}(\rho_1)J^{b_2}(\rho_2)\ldots J^{b_n}(\rho_n)w^{L_0}\right)$$

that can be established using the cyclic properties of the trace and the algebra (6.2), from which it follows that

$$[J_m^a, J^b(z)] = iz^m f^{abc}J^c(z) + kmz^{m-1}\delta^{ab}.$$

For example, in this way it follows that

$$\begin{aligned}
&\mathrm{tr}\left(J_m^a J^{a_1}(\rho_1)\ldots J^{a_n}(\rho_n)w^{L_0}\right)(1-w^m)\\
&= i\sum_{j=1}^{n} f^{aa_j a_j'}\mathrm{tr}\left(J^{a_1}(\rho_1)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a_j'}(\rho_j)J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)\right)\rho_j^m\\
&\quad +km\sum_{j=1}^{n}\delta^{aa_j}\mathrm{tr}\left(J^{a_1}(\rho_1)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)\right)\rho_j^{m-1}
\end{aligned} \tag{6.12}$$

and, hence,

$$\begin{aligned}
&\mathrm{tr}\left(J^a(\rho)J^{a_1}(\rho_1)\ldots J^{a_n}(\rho_n)w^{L_0}\right) = \rho^{-1}\mathrm{tr}\left(J_0^a J^{a_1}(\rho_1)\ldots J^{a_n}(\rho_n)w^{L_0}\right)\\
&+i\sum_{j=1}^{n}\frac{\Delta_1(\nu_j-\nu,\tau)}{\rho}f^{aa_j}_{a_j'}\mathrm{tr}\Big(J^{a_1}(\rho_1)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a_j'}(\rho_j)J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)w^{L_0}\Big)\\
&+k\sum_{j=1}^{n}\frac{\Delta_2(\nu_j-\nu,\tau)}{\rho\rho_j}\delta^{aa_j}\mathrm{tr}\left(J^{a_1}(\rho_1)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)w^{L_0}\right)
\end{aligned} \tag{6.13}$$

where

$$\Delta_1(\nu,\tau) = \sum_{m\neq 0}\frac{e^{2\pi i m\nu}}{1-w^m} = \sum_{m=1}^{\infty}\frac{e^{2\pi i m\nu}-w^m e^{-2\pi i m\nu}}{1-w^m} = \frac{i}{2\pi}\frac{\theta_1'(\nu)}{\theta_1(\nu)}-\frac{1}{2}, \tag{6.14}$$

$$\Delta_2(\nu,\tau) = \sum_{m\neq 0}\frac{me^{2\pi i m\nu}}{1-w^m} = \sum_{m=1}^{\infty}m\,\frac{e^{2\pi i m\nu}+w^m e^{-2\pi i m\nu}}{1-w^m} = \frac{1}{2\pi i}\Delta_1'(\nu,\tau). \tag{6.15}$$

These functions relate to the Weierstrass elliptic functions by

$$\mathcal{P}(\nu,\tau) = -4\pi^2\Delta_2(\nu,\tau)-2\eta(\tau), \qquad \eta(\tau) = -\frac{1}{6}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}, \tag{6.16}$$

$$\zeta(\nu,\tau) = -2\pi i\Delta_1(\nu,\tau)-\pi i+2\eta(\tau)\nu, \qquad \mathcal{P}(\nu,\tau) = -\zeta'(\nu,\tau). \tag{6.17}$$

Using (6.13) and similar relations we can establish general formulae for the two-, three- and four-point one-loop functions.

We write the zero-point (partition) function as

$$\mathrm{tr}\left(w^{L_0}\right) \equiv \chi(\tau). \tag{6.18}$$

Because it is isotropic, we can write

$$\mathrm{tr}\left(J_0^a J_0^b w^{L_0}\right) = \delta^{ab}\chi^{(2)}(\tau) \qquad \text{where} \qquad \chi^{(2)}(\tau) = \frac{1}{\dim G}\mathrm{tr}\left(J_0^a J_0^a w^{L_0}\right).$$

Then the general form of the two-point current algebra loop is

$$\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)w^{L_0}) = \frac{\delta^{ab}k\chi(\tau)}{\rho_1\rho_2}\left[\chi_F(\nu_2-\nu_1,\tau)^2 + f(\tau)\right], \tag{6.19}$$

where

$$f(\tau) = \frac{\chi^{(2)}(\tau)}{k\chi(\tau)} + \frac{\theta_3''(0,\tau)}{4\pi^2\theta_3(0,\tau)}. \tag{6.20}$$

Noting that we can write

$$\mathrm{tr}\left(J_0^a J_0^b J_0^c w^{L_0}\right) = \tfrac{1}{2}\mathrm{tr}\left([J_0^a, J_0^b]J_0^c w^{L_0}\right) + \tfrac{1}{2}\mathrm{tr}\left(\{J_0^a, J_0^b\}J_0^c w^{L_0}\right)$$

we have

$$\mathrm{tr}\left(J_0^a J_0^b J_0^c w^{L_0}\right) = \tfrac{1}{2}if^{abc}\chi^{(2)}(\tau) + \tfrac{1}{2}d^{abc}\chi^{(3)}(\tau),$$

where d^{abc} is a totally symmetric isotropic tensor, which may vanish, as it does for SU(2), but not SU(3). Among the simple Lie groups, only SU(n) with $n \geq 3$ has a symmetric invariant tensor of order 3 [26]. Then the general form of the three-point current algebra loop is

$$\begin{aligned}&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)w^{L_0})\\ &\quad= \frac{ikf^{abc}\chi(\tau)}{\rho_1\rho_2\rho_3}\left\{\chi_F^{21}\chi_F^{32}\chi_F^{13} - \frac{i}{2\pi}\left(\zeta^{21}+\zeta^{32}+\zeta^{13}\right)f(\tau)\right\} + \frac{d^{abc}\chi^{(3)}(\tau)}{2\rho_1\rho_2\rho_3},\end{aligned} \tag{6.21}$$

where $\zeta^{ij} = \zeta(\nu_j - \nu_i)$. The recurrence relations do not manifestly maintain the permutation symmetry of the loop amplitudes but the final result is necessary symmetric and can be put into this form.

The general form of the four-point loop can be put into the symmetric form

$$\begin{aligned}&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})\rho_1\rho_2\rho_3\rho_4\\
&\quad= \Big\{\delta^{ab}\delta^{cd}\left(k^2\chi(\tau)\,[(\chi_F^{12})^2+f(\tau)][(\chi_F^{34})^2+f(\tau)] - \chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad\quad+\delta^{ac}\delta^{bd}\left(k^2\chi(\tau)\,[(\chi_F^{13})^2+f(\tau)][(\chi_F^{24})^2+f(\tau)] - \chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad\quad+\delta^{ad}\delta^{bc}\left(k^2\chi(\tau)\,[(\chi_F^{14})^2+f(\tau)][(\chi_F^{23})^2+f(\tau)] - \chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad\quad+\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_{\mathbf{S}}\\
&\quad\quad-\frac{1}{96}\left(\sigma^{abcd}+\sigma^{adcb}+\sigma^{acdb}+\sigma^{abdc}+\sigma^{adbc}+\sigma^{acbd}\right)\chi(\tau)\theta_2^4(0,\tau)\theta_4^4(0,\tau)\\
&\quad\quad\left(\sigma^{abcd}+\sigma^{adcb}\right)\frac{1}{2}\chi(\tau)\,\Big\{\chi_F^{12}\chi_F^{23}\chi_F^{34}\chi_F^{41}\\
&\quad\quad\quad-\frac{1}{16\pi^2}f(\tau)\left(\frac{\mathcal{P}_{24}'-\mathcal{P}_{32}'}{\mathcal{P}_{24}-\mathcal{P}_{32}}\right)\left(\frac{\mathcal{P}_{24}'-\mathcal{P}_{41}'}{\mathcal{P}_{24}-\mathcal{P}_{41}}\right) + \frac{1}{4\pi^2}f(\tau)\mathcal{P}_{24}\Big\}\end{aligned}$$

$$
\begin{aligned}
&-\left(\sigma^{acdb}+\sigma^{abdc}\right)\frac{1}{2}\chi(\tau)\Big\{\chi_F^{13}\chi_F^{34}\chi_F^{42}\chi_F^{21}\\
&\qquad -\frac{1}{16\pi^2}f(\tau)\left(\frac{\mathcal{P}'_{24}-\mathcal{P}_{32}}{\mathcal{P}_{24}-\mathcal{P}_{32}}\right)\left(\frac{\mathcal{P}'_{21}-\mathcal{P}'_{32}}{\mathcal{P}_{21}-\mathcal{P}_{32}}\right)+\frac{1}{4\pi^2}f(\tau)\mathcal{P}_{32}\Big\}\\
&-\left(\sigma^{adbc}+\sigma^{acbd}\right)\ \frac{1}{2}\chi(\tau)\Big\{\chi_F^{14}\chi_F^{42}\chi_F^{23}\chi_F^{31}\\
&\qquad -\frac{1}{16\pi^2}f(\tau)\left(\frac{\mathcal{P}'_{24}-\mathcal{P}'_{32}}{\mathcal{P}_{24}-\mathcal{P}_{32}}\right)\left(\frac{\mathcal{P}'_{14}-\mathcal{P}'_{31}}{\mathcal{P}_{14}-\mathcal{P}_{31}}\right)+\frac{1}{4\pi^2}f(\tau)\mathcal{P}_{34}\Big\}\\
&-\frac{i}{4\pi}\chi^{(3)}(\tau)\Big[\sigma^{abcd}\left(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14}\right)+\sigma^{adcb}\left(\zeta^{41}+\zeta^{34}+\zeta^{23}+\zeta^{12}\right)\\
&\qquad +\sigma^{acdb}\left(\zeta^{31}+\zeta^{43}+\zeta^{24}+\zeta^{12}\right)+\sigma^{abdc}\left(\zeta^{21}+\zeta^{42}+\zeta^{34}+\zeta^{13}\right)\\
&\qquad +\sigma^{adbc}\left(\zeta^{41}+\zeta^{24}+\zeta^{32}+\zeta^{13}\right)+\sigma^{acbd}\left(\zeta^{31}+\zeta^{23}+\zeta^{42}+\zeta^{14}\right)\Big]\Big\}, \qquad (6.22)
\end{aligned}
$$

where $\mathcal{P}_{ij}=\mathcal{P}(\nu_j-\nu_i,\tau)$, $\mathcal{P}'_{ij}=\mathcal{P}'(\nu_j-\nu_i,\tau)$, σ^{abcd} is given by (6.11) and $\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_{\mathbf{S}}$ is the symmetrization of the trace $\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})$ over permutations of a,b,c,d.

We will now specialize to the case of a general representation of SU(2). In this case,

$$
\begin{aligned}
&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})\rho_1\rho_2\rho_3\rho_4\\
&=\Big\{\delta^{ab}\delta^{cd}\left(k^2\chi(\tau)\left[(\chi_F^{12})^2+f(\tau)\right]\left[(\chi_F^{34})^2+f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad+\delta^{ac}\delta^{bd}\left(k^2\chi(\tau)\left[(\chi_F^{13})^2+f(\tau)\right]\left[(\chi_F^{24})^2+f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad+\delta^{ad}\delta^{bc}\left(k^2\chi(\tau)\left[(\chi_F^{14})^2+f(\tau)\right]\left[(\chi_F^{23})^2+f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad-[\sigma^{abcd}+\sigma^{acdb}+\sigma^{adbc}]\left(\frac{1}{48}\theta_2^4(0,\tau)\theta_4^4(0,\tau)-\frac{1}{6}\frac{\chi^{(4)}(\tau)}{k\chi(\tau)}\right)\chi(\tau)\\
&\quad-\sigma^{abcd}\ \chi(\tau)\Big\{\chi_F^{12}\chi_F^{23}\chi_F^{34}\chi_F^{41}+\frac{f(\tau)}{8\pi^2}\Big[\mathcal{P}_{13}+\mathcal{P}_{24}\\
&\qquad-\left(\zeta^{13}+\zeta^{32}+\zeta^{21}\right)\left(\zeta^{13}+\zeta^{34}+\zeta^{41}\right)-\left(\zeta^{24}+\zeta^{41}+\zeta^{12}\right)\left(\zeta^{24}+\zeta^{43}+\zeta^{32}\right)\Big]\Big\}\\
&\quad-\sigma^{acdb}\ \chi(\tau)\Big\{\chi_F^{13}\chi_F^{34}\chi_F^{42}\chi_F^{21}+\frac{f(\tau)}{8\pi^2}\Big[\mathcal{P}_{14}+\mathcal{P}_{23}\\
&\qquad-\left(\zeta^{14}+\zeta^{42}+\zeta^{21}\right)\left(\zeta^{14}+\zeta^{43}+\zeta^{31}\right)-\left(\zeta^{23}+\zeta^{31}+\zeta^{12}\right)\left(\zeta^{23}+\zeta^{34}+\zeta^{42}\right)\Big]\Big\}\\
&\quad-\sigma^{adbc}\ \chi(\tau)\Big\{\chi_F^{14}\chi_F^{42}\chi_F^{23}\chi_F^{31}+\frac{f(\tau)}{8\pi^2}\Big[\mathcal{P}_{12}+\mathcal{P}_{34} \qquad (6.23)\\
&\qquad-\left(\zeta^{12}+\zeta^{23}+\zeta^{31}\right)\left(\zeta^{12}+\zeta^{24}+\zeta^{41}\right)-\left(\zeta^{34}+\zeta^{41}+\zeta^{13}\right)\left(\zeta^{34}+\zeta^{42}+\zeta^{23}\right)\Big]\Big\}\Big\}.
\end{aligned}
$$

7. Twistor string loop

We now assemble the parts for the one-loop MHV gluon amplitude of the twistor string. The fact that the twistor string has delta function vertices leads to the form for the final integral that is a simple product of the loop for the twistor fields and the current algebra loop. We have provided several equal expressions for the twistor field loop $\mathcal{A}_{n,2}^{\mathrm{loop}}$ in (5.16), (5.18) - (5.22). These forms were given for particles $1,2$ having negative helicity, and we saw that the expressions naturally divide into two factors: a piece equal to the kinematic factor of the tree amplitude multiplied by a function of s and t. For the four-gluon amplitude, if we consider the form given in (5.20), we have

$$\mathcal{A}_{4,2}^{\rm loop} = -\frac{\langle 1,2\rangle^4\delta^4(\Sigma\pi_r\bar{\pi}_r)}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,1\rangle}\frac{s}{t}\int\delta\left(\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_4)(\xi_3-\xi_2)}+\frac{s}{t}\right)\mathcal{A}_4^{J^A}\eta(\tau)^4\prod_{r=1}^4\rho_r d\nu_r\frac{d\alpha_0 d\tau}{2\pi{\rm Im}\tau},$$

where we can take $\xi_r = \theta_3(2\hat{\nu}_r, 2\tau)/\theta_2(2\hat{\nu}_r, 2\tau)$, $\hat{\nu}_r = \nu_r + i\alpha_0/4\pi$ and $\mathcal{A}_4^{J^A}$ is given in (6.22) for a general compact Lie group. We expect the gluon loop amplitude $\mathcal{A}_{\rm loop}$ to be related to the field theory loop amplitude for gluons in $N=4$ Yang Mills theory coupled to $N=4$ conformal supergravity. It is believed this field theory requires a dimension four gauge group to avoid anomalies [20]. We hope that a better understanding of that may follow from further analysis of the loop expressions computed in this paper.

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A. Gauge potentials

An example of a potential on S^2, for which $A_z = A_{\bar{z}} = 0$, is

$$\begin{aligned}
\mathcal{A}_z^< &= -\frac{in\bar{z}}{1+z\bar{z}}, & \tilde{\mathcal{A}}_z^< &= 0,\\
\mathcal{A}_z^> &= \frac{in}{(1+z\bar{z})z}, & \tilde{\mathcal{A}}_z^> &= 0,\\
\mathcal{A}_{\bar{z}}^< &= 0, & \tilde{\mathcal{A}}_{\bar{z}}^< &= -\frac{inz}{1+z\bar{z}},\\
\mathcal{A}_{\bar{z}}^> &= 0, & \tilde{\mathcal{A}}_{\bar{z}}^> &= \frac{in}{(1+z\bar{z})\bar{z}}.
\end{aligned}$$

Then $\mathcal{A}_\mu^> - \mathcal{A}_\mu^< = -ig^{-1}\partial_\mu g$, $\quad\tilde{\mathcal{A}}_\mu^> - \tilde{\mathcal{A}}_\mu^< = -i\tilde{g}^{-1}\partial_\mu\tilde{g}$ for $g = z^{-n}$, $\tilde{g} = \bar{z}^{-n}$.

And an example of a potential on T^2, for which $A_z = A_{\bar{z}} = 0$, is

$$\mathcal{A}_z(z,\bar{z}) = \frac{i\pi n}{{\rm Im}\tau}(z-\bar{z}), \qquad \tilde{\mathcal{A}}_z(z,\bar{z}) = 0,$$

$$\mathcal{A}_{\bar{z}}(z,\bar{z}) = 0, \qquad \tilde{\mathcal{A}}_{\bar{z}}(z,\bar{z}) = -\frac{i\pi n}{{\rm Im}\tau}(z-\bar{z}),$$

Then

$$\begin{aligned}
\mathcal{A}_\mu(z+a,\bar{z}+\bar{a}) - \mathcal{A}_\mu(z,\bar{z}) &= -ig_a^{-1}(\bar{z})\partial_\mu g_a(\bar{z}),\\
\tilde{\mathcal{A}}_\mu(z+a,\bar{z}+\bar{a}) - \tilde{\mathcal{A}}_\mu(z,\bar{z}) &= -i\tilde{g}_a^{-1}(z)\partial_\mu\tilde{g}_a(z)
\end{aligned}$$

for

$$\begin{aligned}
g_a(z) &= e^{-\frac{\pi n(a-\bar{a})}{{\rm Im}\tau}(z+\frac{a}{2})+i\pi n m_1 n_1 + i\eta_a},\\
\tilde{g}_a(\bar{z}) &= e^{\frac{\pi n(a-\bar{a})}{{\rm Im}\tau}(\bar{z}+\frac{\bar{a}}{2})-i\pi n m_1 n_1 - i\bar{\eta}_a}.
\end{aligned}$$

B. Twistors

A Riemann surface world sheet, which we use in this paper, corresponds to complex target space fields Z^I that transform under the conformal group SU(2,2). The spacetime metric is $\eta = \mathrm{diag}(1,-1,-1,-1)$, and $\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1$. The coordinates are $x = x^0 - \mathbf{x}\cdot\sigma$, with $x^\dagger = x$, and $\det x = x^\mu x_\mu$. Then

$$x \mapsto (ax+b)(cx+d)^{-1}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2,2)$.

The two 2-spinors $(\pi_a, \omega^{\dot a})$ define complex 2-planes in complexified Minkowski space by $\pi = x\omega$. Under a conformal transformation, the twistor

$$\begin{pmatrix} \pi \\ \omega \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi \\ \omega \end{pmatrix}.$$

Real Lorentz transformations are given by $x \mapsto axa^\dagger$, $\pi \mapsto a\pi$, $\omega \mapsto d\omega$ where $d = (a^\dagger)^{-1}$ and $a, d \in \mathrm{SL}(2,\mathbb{C})$. For any vector p^μ, we can write

$$\mathrm{p} = p^\mu \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \equiv (\mathrm{p}_{a\dot a}), \qquad \mathrm{p}^\dagger = \mathrm{p},$$

where $\epsilon^{ab}\epsilon^{\dot a\dot b}\mathrm{p}_{a\dot a}\mathrm{p}_{b\dot b} = 2\det(\mathrm{p}) = 2p_\mu p^\mu \equiv 2p^2$, and $\epsilon^{ab}\epsilon^{\dot a\dot b}\mathrm{p}_{a\dot a}\mathrm{q}_{b\dot b} = 2p\cdot q$. Then real Lorentz transformations are given by $\mathrm{p} \mapsto a\mathrm{p}a^\dagger$, $a \in \mathrm{SL}(2,\mathbb{C})$; and the complex Lorentz transformations are $\mathrm{p} \mapsto a\mathrm{p}b$, $a, b \in \mathrm{SL}(2,\mathbb{C})$.

If $p^2 = 0$, we can write $\mathrm{p} = \lambda\bar\lambda^T$, i.e. $\mathrm{p}_{a\dot a} = \lambda_a\bar\lambda_{\dot a}$. Similarly, if $q^2 = 0$, $\mathrm{q}_{b\dot b} = \mu_b\bar\mu_{\dot b}$, then $2p\cdot q = \epsilon^{ab}\lambda_a\mu_b\,\epsilon^{\dot a\dot b}\bar\lambda_{\dot a}\bar\mu_{\dot b} = \langle\lambda,\mu\rangle\,[\bar\lambda,\bar\mu]$, where $\langle\lambda,\mu\rangle = \epsilon^{ab}\lambda_a\mu_b$ and $[\bar\lambda,\bar\mu] = \epsilon^{\dot a\dot b}\bar\lambda_{\dot a}\bar\mu_{\dot b}$. If $\sum_{r=1}^n p_r = 0$, with $p_{ra\dot a} = \pi_{ra}\bar\pi_{r\dot a}$, then $\sum_{r=1}^n \pi_{ra}\bar\pi_{r\dot a} = 0$; and $\sum_{r\neq s}\langle sr\rangle\bar\pi_{r\dot a} = 0$ and $\sum_{r\neq s}[sr]\pi_{ra} = 0$, where $[rs] = [\bar\pi_r, \bar\pi_s]$, and $\langle rs\rangle = \langle\pi_r, \pi_s\rangle$. e.g., for $n = 3$, $\pi_{1a}/\pi_{2a} = -[2,3]/[1,3]$.

Under conformal transformations,

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \qquad \begin{pmatrix} \psi_\lambda \\ \psi_\mu \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_\lambda \\ \psi_\mu \end{pmatrix},$$

where $\psi_\lambda = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ and $\psi_\mu = \begin{pmatrix} \psi^3 \\ \psi^4 \end{pmatrix}$.

Polarizations. To compute the polarization dependence of the amplitudes as in (4.6), we note that $s_{\dot a}$ and $\bar s_a$ are reference vectors that scale as the momentum: $s_{\dot a}, \pi^a \sim u$ scale up and $\bar s_a, \bar\pi^{\dot a} \sim u^{-1}$ scale down. The polarization vector $\epsilon_{a\dot a\,r}$ does not scale, so $A_{-\,r}$ goes as u^{-2} and $A_{+\,r}$ as u^2. Hence the vertex operator $(\pi^1)^{-2}\left[A_{+\,r} + (\frac{\pi^1}{\lambda^1})^4\,\psi^1\psi^2\psi^3\psi^4\,A_{-\,r}\right]$ does not scale. Then

$$\begin{aligned} &\epsilon_1^- \cdot \epsilon_2^+\epsilon_3^+ \cdot p_1 + \epsilon_2^+ \cdot \epsilon_3^+\epsilon_1^- \cdot p_2 + \epsilon_3^+ \cdot \epsilon_1^-\epsilon_2^+ \cdot p_3 \\ &= A_{-1}A_{+2}A_{+3}\;(\pi_1^b\bar s_2^a\bar s_{3b}\pi_{1a}\bar\pi^{3\dot b}\bar\pi_{2\dot b}) = A_{-1}A_{+2}A_{+3}\;\tfrac{[23]^3}{[12][31]}\,. \end{aligned}$$

To derive this we use momentum conservation $\sum_{r=1}^{3} \pi_{ra}\bar{\pi}_{r\dot{a}} = 0$, and properties of the Penrose spinors such as $\sum_{a=1}^{2} \pi_r^a \pi_{ra} = 0$, where $\pi_{ra} = \epsilon_{ab}\pi_r^b$ and $\pi_r^a = \epsilon^{ab}\pi_{rb}$. In particular, consider $\bar{s}_{3b} \sum_{r=1}^{3} \pi_r^b \bar{\pi}_r^{\dot{b}} = 0$ to find $\bar{s}_{3b}\pi_1^b = [23]/[12]$, and similarly $\bar{s}_2^a \pi_{1a} = [23]/[13]$.

C. Trace calculations

Bosonic trace. To calculate the M-point trace, as in (3.18) let

$$\Phi(\omega, k) = \mathrm{tr}\left(e^{q_0} e^{ik_1 Z_0} e^{q_0} e^{ik_2 Z_0} \dots e^{q_0} e^{ik_d Z_0} u^{a_0} e^{i\omega_1 Z(\rho_1)} e^{i\omega_2 Z(\rho_2)} \dots e^{i\omega_M Z(\rho_M)} w^{L_0}\right),$$

so

$$\Phi(0, k) = \prod_{i=1}^{d} \delta(k_i),$$

because only the states with no non-zero modes contribute to this trace. Let

$$\Phi_n(\omega, k) = \mathrm{tr}\left(e^{q_0} e^{ik_1 Z_0} e^{q_0} e^{ik_2 Z_0} \dots e^{q_0} e^{ik_d Z_0} u^{a_0} Z_n e^{i\omega_1 Z(\rho_1)} e^{i\omega_2 Z(\rho_2)} \dots e^{i\omega_M Z(\rho_M)} w^{L_0}\right),$$

so

$$\partial_{k_1}\Phi = iu\Phi_{1-d}, \quad \dots \quad \partial_{k_{d-m}}\Phi = iu\Phi_{-m}, \quad \dots \quad \partial_{k_d}\Phi = iu\Phi_0,$$

using

$$e^{q_0} Z_n e^{-q_0} = Z_{n+1}, \qquad u^{a_0} Z_n u^{-a_0} = u^{-1} Z_n.$$

Then, using the cyclic property of trace,

$$\Phi_n = uw^n \Phi_{n-d} = u^{(n+m)/d} w^{(n+d-m)(n+m)/2d} \Phi_{-m}, \quad 0 \le m < d, \quad n+m \text{ a multiple of } d$$

and

$$\begin{aligned}
\partial_{\omega_j}\Phi = i\sum_{n=-\infty}^{\infty} \Phi_n \rho_j^{-n} &= i\sum_{i=0}^{d-1} \sum_{n=-\infty}^{\infty} \Phi_{dn-i} \rho_j^{-dn+i} \\
&= i\sum_{i=0}^{d-1} \Phi_{-i} \sum_{n=-\infty}^{\infty} u^n w^{\frac{1}{2}n(d(n+1)-2i)} \rho_j^{-dn+i} \\
&= i\sum_{i=0}^{d-1} \Phi_{-i} \sum_{n=-\infty}^{\infty} w^{\frac{1}{2}(n-1)(dn-2i)} u^{n-i/d} \left(\tfrac{\rho_j}{w}\right)^{-dn+i} u^{i/d} \\
&= i\sum_{i=0}^{d-1} \Phi_{-i} F_{d-i}^d (u^{-1/d}\rho_j, w) u^{i/d} \\
&= \sum_{n=1}^{d} F_n^d (u^{-1/d}\rho_j, w) u^{-n/d} \partial_{k_n}\Phi
\end{aligned}$$

where

$$F_i^d(\rho_j, w) = \sum_{n=-\infty}^{\infty} w^{\frac{1}{2}(n-1)(d(n-2)+2i)} \left(\frac{\rho_j}{w}\right)^{d(1-n)-i}$$

So Φ depends on k_i, ω_j, $1 \le i \le d, 1 \le j \le M$, through

$$u^{i/d} k_i + \sum_{j=1}^{M} F_i^d(\hat{\rho}_j, w)\omega_j, \qquad \text{where } \hat{\rho}_j = u^{-1/d}\rho_j,$$

so that

$$\Phi(\omega, k) = u^{(d+1)/2} \prod_{i=1}^{d} \delta\left(u^{i/d} k_i + \sum_{j=1}^{M} F_i^d(\hat{\rho}_j, w)\omega_j\right)$$

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and the M point function

$$
\begin{aligned}
&\mathrm{tr}\left(e^{dq_0}u^{a_0}e^{i\omega_1 Z(\rho_1)}e^{i\omega_2 Z(\rho_2)}\dots e^{i\omega_M Z(\rho_M)}w^{L_0}\right)\\
&= \Phi(\omega,0)\\
&= u^{(d+1)/2}\prod_{i=1}^{d}\delta\left(\sum_{j=1}^{M}F_i^d(\hat\rho_j,w)\omega_j\right) \qquad \text{(C.1)}\\
&= u^{(d+1)/2}\begin{vmatrix} F_1^d(\hat\rho_1,w) & F_1^d(\hat\rho_2,w) & \dots & F_1^d(\hat\rho_d,w)\\ F_2^d(\hat\rho_1,w) & F_2^d(\hat\rho_2,w) & \dots & F_2^d(\hat\rho_d,w)\\ . & . & \dots & .\\ . & . & \dots & .\\ F_d^d(\hat\rho_1,w) & F_d^d(\hat\rho_2,w) & \dots & F_d^d(\hat\rho_d,w)\end{vmatrix}^{-1}\prod_{m=1}^{d}\delta(\omega_m), \qquad \text{(C.2)}
\end{aligned}
$$

where the last equality (C.2) holds provided that $M = d$.

Comparison with bosonic tree amplitudes. We can compare these results for corresponding results for tree amplitudes,

$$\langle 0|e^{dq_0}e^{i\omega_M Z(\rho_M)}\dots e^{i\omega_0 Z_0}|0\rangle.$$

We start with

$$\langle 0|e^{dq_0}e^{ik_d Z_{-d}}\dots e^{ik_0 Z_0)}|0\rangle = \prod_{i=0}^{d}\delta(k_i).$$

Let

$$\Phi = \langle 0|e^{dq_0}\prod_{i=0}^{d}e^{ik_i Z_{-i}}\prod_{j=0}^{M}e^{i\omega_j Z(\rho_j)}|0\rangle$$

$$
\begin{aligned}
\partial_{\omega_j}\Phi &= i\textstyle\sum_{k=0}^{d}\langle 0|e^{dq_0}\prod_{i=0}^{d}e^{ik_i Z_{-i}}\prod_{j=0}^{M}e^{i\omega_k Z(\rho_j)}Z_{-k}|0\rangle\rho_j^k\\
&= \textstyle\sum_{i=0}^{d}\rho_j^i\partial_{k_i}\Phi.
\end{aligned}
$$

These provide $M+1$ linear partial differential equations in the $M+d+2$ variables k,ω, implying that Φ depends only on the $d+1$ variables

$$k_i + \sum_{j=0}^{M}\rho_j^i\omega_j$$

so that

$$\Phi = \prod_{i=0}^{d}\delta\left(k_i + \sum_{j=0}^{M}\rho_j^i\omega_j\right). \qquad \text{(C.3)}$$

Thus

$$\begin{aligned}\langle 0|e^{dq_0}e^{i\omega_M Z(\rho_M)}\dots e^{i\omega_0 Z(\rho_0)}|0\rangle &= \prod_{i=0}^{d}\delta\left(\sum_{j=0}^{M}\rho_j^i\omega_j\right)\\ &= \begin{vmatrix} 1 & 1 & \dots & 1\\ \rho_0 & \rho_1 & \dots & \rho_d\\ . & . & \dots & .\\ . & . & \dots & .\\ \rho_0^d & \rho_1^d & \dots & \rho_d^d\end{vmatrix}^{-1}\prod_{j=0}^{d}\delta(\omega_j)\\ &= \prod_{0\le i<j\le d}(\rho_i-\rho_j)^{-1}\prod_{j=0}^{d}\delta(\omega_j), \end{aligned} \tag{C.4}$$

provided that $M = d$.

Fermionic tree amplitudes. Starting from

$$\langle 0|e^{dq_0}Z_{-d}\dots Z_{-1}Z_0|0\rangle = 1,$$

so that

$$\langle 0|e^{dq_0}Z_{-n_d}\dots Z_{-n_1}Z_{-n_0}|0\rangle = \epsilon_{n_d\dots n_1 n_0},$$

where $\epsilon_{n_d\dots n_1 n_0}$ is totally antisymmetric and nonzero only if $(n_d,\dots,n_1,n_0)$ is a permutation of $(d,\dots,1,0)$ with $\epsilon_{d\dots 10} = 1$. Then

$$\begin{aligned}\langle 0|e^{dq_0}c(\rho_d)\dots c(\rho_1)c(\rho_0)|0\rangle &= \sum_{n_j=-\infty}^{\infty}\rho_d^{n_d}\dots\rho_1^{n_1}\rho_0^{n_0}\langle 0|e^{dq_0}Z_{-n_d}\dots Z_{-n_1}Z_{-n_0}|0\rangle\\ &= \sum_{n_j=0}^{d}\epsilon_{n_d\dots n_1 n_0}\rho_d^{n_d}\dots\rho_1^{n_1}\rho_0^{n_0}\\ &= \begin{vmatrix} 1 & 1 & \dots & 1\\ \rho_0 & \rho_1 & \dots & \rho_d\\ . & . & \dots & .\\ . & . & \dots & .\\ \rho_0^d & \rho_1^d & \dots & \rho_d^d\end{vmatrix}\\ &= \prod_{0\le i<j\le d}(\rho_i-\rho_j), \end{aligned} \tag{C.5}$$

which can be compared with (C.2).

Fermionic loop amplitudes. Consider

$$\mathrm{tr}\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z(\rho_d)\dots Z(\rho_2)Z(\rho_1)w^{L_0}\right),$$

noting that

$$\begin{aligned}&\mathrm{tr}\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z_{-d}\dots Z_{-2}Z_{-1}w^{L_0}\right)\\ &\qquad = u^{-d}(-1)^{Nd}\mathrm{tr}\left(e^{dq_0}Z_{-d}\dots Z_{-2}Z_{-1}u^{a_0}(-1)^{Na_0}w^{L_0}\right)\\ &\qquad = u^{-d}(-1)^{Nd}\langle 0|e^{dq_0}Z_{-d}\dots Z_{-2}Z_{-1}u^{a_0}(-1)^{Na_0}w^{L_0}|0\rangle\\ &\qquad = u^{-d}(-1)^{Nd},\end{aligned}$$

where N is an integer, because again only the states with no non-zero modes contribute to this trace.

If $\varphi(Z)$ denotes an arbitrary sum of products of the Z_m each of length $d-1$,

$$\Phi_n = \mathrm{tr}\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z_n\varphi(Z)w^{L_0}\right) = (-1)^{N+d-1}uw^n\Phi_{n-d}$$

using the cyclic property of trace.

So, as for the bosonic traces,

$$\Phi_{dn-m} = \eta^n w^{\frac{1}{2}n(d(n+1)-2m)}\Phi_{-m}, \qquad 0 \le m < d$$

where $\eta = (-1)^{N+d-1}u$ and, again,

$$\begin{aligned} &tr\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z(\rho_j)\varphi(Z)w^{L_0}\right) \\ &\qquad = \textstyle\sum_{n=-\infty}^{\infty}\Phi_n\rho_j^{-n} = \sum_{m=0}^{d-1}\sum_{n=-\infty}^{\infty}\Phi_{dn-m}\rho_j^{-dn+m} \\ &\qquad = \textstyle\sum_{m=0}^{d-1}\Phi_{-m}\sum_{n=-\infty}^{\infty}\epsilon^n u^n w^{\frac{1}{2}n(d(n+1)-2m)}\rho_j^{-dn+m} \\ &\qquad = \textstyle\sum_{m=0}^{d-1}\Phi_{-m}\sum_{n=-\infty}^{\infty}\epsilon^n w^{\frac{1}{2}(n-1)(dn-2m)}u^{n-m/d}\left(\frac{\rho_j}{w}\right)^{-dn+m}u^{m/d} \\ &\qquad = \textstyle\sum_{m=0}^{d-1}u^{m/d}\mathrm{tr}\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z_{-m}\varphi(Z)w^{L_0}\right)F^{d\epsilon}_{d-m}(u^{-1/d}\rho_j, w) \end{aligned}$$

where $\epsilon = (-1)^{N+d-1}$ and

$$F^{d\epsilon}_m(\rho, w) = \sum_{n=-\infty}^{\infty}\epsilon^n w^{\frac{1}{2}(n-1)(d(n-2)+2m)}\left(\frac{\rho}{w}\right)^{d(n-1)-m}$$

Writing $\hat{\rho}_j = u^{-1/d}\rho_j$, it follows that

$$\begin{aligned} &\mathrm{tr}\left(e^{dq_0}u^{a_0}(-1)^{Na_0}Z(\rho_d)\dots Z(\rho_2)Z(\rho_1)w^{L_0}\right) \\ &= \sum_{m_j=1}^{d}\mathrm{tr}\Big(e^{dq_0}u^{a_0}(-1)^{Na_0}Z_{-m_d}\dots Z_{-m_2}Z_{-m_1}w^{L_0}\Big)\prod_{j=1}^{d}u^{(m_j-1)/d}F^{d\epsilon}_{d-m_j+1}(\hat{\rho}_j, w) \\ &= u^{-(d+1)/2}(-1)^{Nd}\sum_{m_j=1}^{d}\prod_{j=1}^{d}\epsilon_{m_d\dots m_2m_1}F^{d\epsilon}_{d-m_j+1}(\hat{\rho}_j, w) \\ &= u^{-(d+1)/2}\sum_{m_j=1}^{d}\prod_{j=1}^{d}\epsilon_{m_d\dots m_2m_1}F^{d}_{d-m_j+1}(\hat{\rho}_j, w), \end{aligned} \tag{C.6}$$

provided that $N=0$ when d is odd and $N=1$ when d is even, because then $\epsilon = 1$ and $F^{d1}_m(\rho, w) = F^d_m(\rho, w)$. i.e. a factor $(-1)^{a_0}$ is included when d is even and no factor of $(-1)^{Na_0}$ is included if d is odd. In particular, in this case, there is a precise cancellation between (C.2) and (C.5) in the case $M = d$. More generally, for $M \neq d$, the overall factors of u cancel and the functions $F^d_m(\hat{\rho}, w)$ involved are the same. We shall use this factor in computing the trace for the twistor string loop, and find agreement with the path integral derivation. Since $[Q, a^i_0] = 0$, it still holds that only states in the cohomology of Q contribute to the trace following the discussion at the end of section 3.

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D. Current algebra loop from recurrence relations

We outline the derivation of the three and four-point current algebra one-loop amplitude from the recurrence relations described in section 6. For the three-point function, from (6.12) and (6.13),

$$\begin{aligned}
&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)w^{L_0}) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1}\Big\{\chi^{(2)}(\tau)\left(\tilde\Delta_1(\nu_{12},\tau)+\tilde\Delta_1(\nu_{23},\tau)+\tilde\Delta_1(\nu_{31},\tau)\right) \\
&\quad +k\chi(\tau)\left(-\Delta_2^1(\nu_{23},\tau)+(\tilde\Delta_1(\nu_{12},\tau)+\tilde\Delta_1(\nu_{31},\tau))\Delta_2(\nu_{23},\tau)\right)\Big\} \\
&\quad +\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1} \\
&\quad \cdot\Big\{\left(\chi^{(2)}(\tau)+k\chi(\tau)\Delta_2(\nu_{23},\tau)\right)\left(\tilde\Delta_1(\nu_{12},\tau)+\tilde\Delta_1(\nu_{23},\tau)+\tilde\Delta_1(\nu_{31},\tau)\right) \\
&\qquad -\tfrac{1}{4\pi i}k\chi(\tau)\partial_\nu\Delta_2(\nu_{23},\tau)\Big\}+\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau)
\end{aligned}$$

where $\nu_{ij}=\nu_j-\nu_i$, and from (6.12) we encounter propagators in addition to those in section 6:

$$\begin{aligned}
\tilde\Delta_1(\nu,\tau) &= \Delta_1(\nu,\tau)+\tfrac12 \\
\Delta_2(\nu,\tau) &= \tfrac{1}{2\pi i}\partial_\nu\Delta_1(\nu,\tau)=\tfrac{1}{4\pi^2}\left(\tfrac{\theta_1'(\nu)}{\theta_1(\nu)}\right)'=\chi_F^2(\nu,\tau)+\tfrac{1}{4\pi^2}\tfrac{\theta_3''(0,\tau)}{\theta_3(0,\tau)} \\
\Delta_1^1(\nu,\tau) &= \textstyle\sum_{m\neq0}\tfrac{e^{2\pi im\nu}w^m}{(1-w^m)^2}=\sum_{m\neq0}\tfrac{e^{2\pi im\nu}}{(1-w^m)^2}-\Delta_1(\nu,\tau) \\
&= \tfrac12\Delta_2(\nu,\tau)-\tfrac12(\tilde\Delta_1(\nu,\tau))^2+\tfrac{1}{12}-\tfrac{1}{24\pi^2}\tfrac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)} \\
\Delta_2^1(\nu,\tau) &= \textstyle\sum_{m\neq0}\tfrac{me^{2\pi im\nu}w^m}{(1-w^m)^2}=\tfrac{1}{2\pi i}\partial_\tau\Delta_1(\nu,\tau)=\tfrac{1}{2\pi i}\partial_\nu\Delta_1^1(\nu,\tau) \\
&= \tfrac{i}{16\pi^3}\tfrac{\theta_1'(\nu,\tau)\theta_1''(\nu,\tau)-\theta_1'''(\nu,\tau)\theta_1(\nu,\tau)}{\theta_1^2(\nu,\tau)}=\tfrac{1}{4\pi i}\partial_\nu\Delta_2(\nu,\tau)-\tilde\Delta_1(\nu,\tau)\Delta_2(\nu,\tau)\,.
\end{aligned}$$

Let

$$\tilde{\mathcal{P}}(\nu)=-4\pi^2\chi_F^2(\nu,\tau)-\theta_3^{-1}(0,\tau)\theta_3''(0,\tau)-4\pi^2\frac{\chi^{(2)}(\tau)}{\chi(\tau)}\,,$$

which depends on the partition function $\chi(\tau)$ and the propagator $\chi^{(2)}(\tau)$ of a given representation. It is related to the standard Weierstrass P function $\mathcal{P}(\nu,\tau)$ by an additive function of τ. Then

$$\begin{aligned}
&\mathrm{tr}\ (J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)w^{L_0}) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1}k\chi(\tau)(-\tfrac{i}{16\pi^3})\Big\{2\tilde{\mathcal{P}}_{23}\left(\zeta^{23}+\zeta^{31}+\zeta^{12}\right)+\tilde{\mathcal{P}}_{23}'\Big\}+\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau).
\end{aligned}$$

Using the Weierstrass addition formulae, which hold for both $\mathcal{P}(\nu,\tau)$ and $\tilde{\mathcal{P}}(\nu,\tau)$:

$$\begin{aligned}
\zeta^{23} &= \zeta^{13}+\zeta^{21}+\tfrac12\tfrac{\mathcal{P}_{13}'-\mathcal{P}_{21}'}{\mathcal{P}_{13}-\mathcal{P}_{21}} \\
\mathcal{P}_{23}' &= \tfrac{\mathcal{P}_{23}[\mathcal{P}_{21}'-\mathcal{P}_{13}']+\mathcal{P}_{21}\mathcal{P}_{13}'-\mathcal{P}_{21}'\mathcal{P}_{13}}{\mathcal{P}_{13}-\mathcal{P}_{21}}\,,
\end{aligned}$$

we can write

$$\begin{aligned}
&\mathrm{tr}\ (J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)w^{L_0}) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1}k\chi(\tau)(-\tfrac{i}{16\pi^3})\Big\{\tilde{\mathcal{P}}_{23}\tfrac{\tilde{\mathcal{P}}_{13}'-\tilde{\mathcal{P}}_{21}'}{\tilde{\mathcal{P}}_{13}-\tilde{\mathcal{P}}_{21}}+\tilde{\mathcal{P}}_{23}'\Big\}+\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1}k\chi(\tau)(-\tfrac{i}{16\pi^3})\Big\{\tfrac{\tilde{\mathcal{P}}_{21}\tilde{\mathcal{P}}_{13}'-\tilde{\mathcal{P}}_{21}'\tilde{\mathcal{P}}_{13}}{\tilde{\mathcal{P}}_{13}-\tilde{\mathcal{P}}_{21}}\Big\}+\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau) \\
&= if^{abc}(\rho_1\rho_2\rho_3)^{-1}k\chi(\tau)\Big\{\chi_F^{21}\chi_F^{32}\chi_F^{13}-\tfrac{i}{2\pi}\left(\zeta^{21}+\zeta^{32}+\zeta^{13}\right)f(\tau)\Big\} \\
&\quad +(\rho_1\rho_2\rho_3)^{-1}\tfrac{d^{abc}}{2\rho_1\rho_2\rho_3}\chi^{(3)}(\tau)
\end{aligned}$$

where the last line follows from standard theta function addition formulae [27]. In a similar way, we compute from the recurrence relation the four-point loop as

$$
\begin{aligned}
&\mathrm{tr}\;(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})\\
&\quad=(\rho_1\rho_2\rho_3\rho_4)^{-1}\Big\{\delta^{ab}\delta^{cd}\Big(\Big[\tfrac{\chi^{(2)}(\tau)}{\chi(\tau)}+k\Delta_2(\nu_{12},\tau)\Big]\left[\chi^{(2)}(\tau)+k\chi(\tau)\Delta_2(\nu_{34},\tau)\right]\\
&\qquad\qquad\qquad\qquad -\chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\Big)\\
&\qquad\qquad +\delta^{ac}\delta^{bd}\Big(\Big[\tfrac{\chi^{(2)}(\tau)}{\chi(\tau)}+k\Delta_2(\nu_{13},\tau)\Big]\left[\chi^{(2)}(\tau)+k\chi(\tau)\Delta_2(\nu_{24},\tau)\right]\\
&\qquad\qquad\qquad\qquad -\chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\Big)\\
&\qquad\qquad +\delta^{ad}\delta^{bc}\Big(\Big[\tfrac{\chi^{(2)}(\tau)}{\chi(\tau)}+k\Delta_2(\nu_{14},\tau)\Big]\left[\chi^{(2)}(\tau)+k\chi(\tau)\Delta_2(\nu_{23},\tau)\right]\\
&\qquad\qquad\qquad\qquad -\chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\Big)\\
&\qquad\qquad +\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})\\
&\qquad\qquad +f_e^{ab}f^{cde}\Big[-\tfrac{1}{2}\chi^{(2)}(\tau)\Delta_1(\nu_{34},\tau)\\
&\qquad\qquad\quad +\Big(\Delta_1^1(\nu_{23},\tau)-\tilde{\Delta}_1^1(\nu_{24},\tau)\Big)\left(\chi^{(2)}(\tau)+k\chi(\tau)\Delta_2(\nu_{34},\tau)\right)\\
&\qquad\qquad\quad -\tilde{\Delta}_1(\nu_{12},\tau)\left[\chi^{(2)}(\tau)(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{34},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\right.\\
&\qquad\qquad\quad \left.+k\chi(\tau)(-\Delta_2^1(\nu_{34},\tau)+(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\Delta_2(\nu_{34},\tau))\right]\Big]\\
&\qquad\qquad +f_e^{ac}f^{bde}\Big[-\chi^{(2)}(\tau)\Delta_1^1(\nu_{34},\tau)+k\chi(\tau)\Delta_3(\nu_{34},\tau)\\
&\qquad\qquad\quad -\chi^{(2)}(\tau)\tilde{\Delta}_1(\nu_{34},\tau)\Delta_1(\nu_{24},\tau)+k\chi(\tau)\Delta_2^1(\nu_{34},\tau)\Delta_1(\nu_{24},\tau)\\
&\qquad\qquad\quad +k\chi(\tau)\Delta_2^1(\nu_{34},\tau)\\
&\qquad\qquad\quad +\tilde{\Delta}_1(\nu_{13},\tau)\left[\chi^{(2)}(\tau)(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{34},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\right.\\
&\qquad\qquad\quad \left.+k\chi(\tau)(-\Delta_2^1(\nu_{34},\tau)+(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\Delta_2(\nu_{34},\tau))\right]\Big]\\
&\qquad\qquad +f_e^{ad}f^{bce}\Big[-\chi^{(2)}(\tau)\Delta_1^1(\nu_{34},\tau)+k\chi(\tau)\Delta_3(\nu_{34},\tau)\\
&\qquad\qquad\quad +\chi^{(2)}(\tau)\tilde{\Delta}_1(\nu_{34},\tau)\Delta_1(\nu_{23},\tau)-k\chi(\tau)\Delta_2^1(\nu_{34},\tau)\Delta_1(\nu_{23},\tau)\\
&\qquad\qquad\quad -\tilde{\Delta}_1(\nu_{14},\tau)\left[\chi^{(2)}(\tau)(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{34},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\right.\\
&\qquad\qquad\quad \left.+k\chi(\tau)(-\Delta_2^1(\nu_{34},\tau)+(\tilde{\Delta}_1(\nu_{23},\tau)+\tilde{\Delta}_1(\nu_{42},\tau))\Delta_2(\nu_{34},\tau))\right]\Big]\\
&\qquad\qquad +\tfrac{i}{2}\chi^{(3)}(\tau)\Big(f_e^{ab}d^{ecd}\Delta_1(\nu_{12})+f_e^{ac}d^{bed}\Delta_1(\nu_{13})+f_e^{ad}d^{bce}\Delta_1(\nu_{14})\\
&\qquad\qquad\qquad +f_e^{bc}d^{aed}\Delta_1(\nu_{23})+f_e^{bd}d^{ace}\Delta_1(\nu_{24})+f_e^{cd}d^{abe}\Delta_1(\nu_{34})\Big)\Big\},
\end{aligned}
$$

where an additional propagator occurs,

$$
\begin{aligned}
\Delta_3(\nu,\tau)&=\textstyle\sum_{m\neq 0}\frac{me^{2\pi i m\nu}w^{2m}}{(1-w^m)^3}=-\tfrac{1}{2}\Delta_2^1(\nu,\tau)+\tfrac{1}{4\pi i}\partial_\tau\Delta_1^1(\nu,\tau)\\
&=-\tfrac{1}{2}\Delta_2^1(\nu,\tau)-\tfrac{1}{2}\tilde{\Delta}_1(\nu,\tau)\Delta_2^1(\nu,\tau)+\tfrac{1}{8\pi i}\partial_\nu\Delta_2^1(\nu,\tau)+\tfrac{i}{96\pi^3}\partial_\tau\left(\tfrac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}\right).
\end{aligned}
$$

This form of the four-point current algebra loop can be expressed in terms of the Weierstrass

$\mathcal{P}$ functions:

$$
\begin{aligned}
&\mathrm{tr}\ (J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})\rho_1\rho_2\rho_3\rho_4 \\
&= \Big\{\delta^{ab}\delta^{cd}\ \left(\tfrac{1}{16\pi^2}k^2\chi(\tau)\,\tilde{\mathcal{P}}_{12}\tilde{\mathcal{P}}_{34} - \chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\right) \\
&\qquad +\delta^{ac}\delta^{bd}\ \left(\tfrac{1}{16\pi^2}k^2\chi(\tau)\,\tilde{\mathcal{P}}_{31}\tilde{\mathcal{P}}_{24} - \chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\right) \\
&\qquad +\delta^{ad}\delta^{bc}\ \left(\tfrac{1}{16\pi^2}k^2\chi(\tau)\,\tilde{\mathcal{P}}_{14}\tilde{\mathcal{P}}_{32} - \chi^{-1}(\tau)(\chi^{(2)}(\tau))^2\right) \\
&\qquad +\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0}) \\
&\qquad +f^{abe}f^{cde}[\tfrac{1}{64\pi^4}k\chi(\tau)\Big\{\tfrac{\mathcal{P}_{32}\mathcal{P}'_{24}-\mathcal{P}'_{32}\mathcal{P}_{24}}{\mathcal{P}_{24}-\mathcal{P}_{32}}\ \left(-\tfrac{\mathcal{P}'_{24}-\mathcal{P}'_{32}}{\mathcal{P}_{24}-\mathcal{P}_{32}} + \tfrac{\mathcal{P}'_{24}+\mathcal{P}'_{14}}{\mathcal{P}_{24}-\mathcal{P}_{14}}\right) \\
&\qquad\qquad +4\tilde{\mathcal{P}}_{34}\tilde{\mathcal{P}}_{32}\Big\} \\
&\qquad\quad +k\chi(\tau)\ \Big\{-\left(\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)}+\tfrac{1}{4\pi^2}\tfrac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right)^2 + \tfrac{1}{48}\theta_2^4(0,\tau)\theta_4^4(0,\tau) + \tfrac{1}{3}\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)} \\
&\qquad\qquad -\tfrac{1}{6}\left(\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)}+\tfrac{1}{4\pi^2}\tfrac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right)\left(\theta_2^4(0,\tau)-\theta_4^4(0,\tau)\right)\ \Big\}] \\
&\qquad +f_e^{ac}f^{bde}[\tfrac{1}{64\pi^4}k\chi(\tau)\,\{\tfrac{\mathcal{P}_{32}\mathcal{P}'_{24}-\mathcal{P}'_{32}\mathcal{P}_{24}}{\mathcal{P}_{24}-\mathcal{P}_{32}}\ \left(\tfrac{\mathcal{P}'_{14}-\mathcal{P}'_{31}}{\mathcal{P}_{14}-\mathcal{P}_{31}} + \tfrac{\mathcal{P}'_{24}-\mathcal{P}'_{32}}{\mathcal{P}_{24}-\mathcal{P}_{32}}\right) \\
&\qquad\quad -4\tilde{\mathcal{P}}_{34}(\tilde{\mathcal{P}}_{24}+\tilde{\mathcal{P}}_{32})\} \\
&\qquad\quad +k\chi(\tau)\ \Big\{2\left(\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)}+\tfrac{1}{4\pi^2}\tfrac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right)^2 - \tfrac{1}{24}\theta_2^4(0,\tau)\theta_4^4(0,\tau) - \tfrac{1}{6}\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)} \\
&\qquad\qquad +\tfrac{1}{3}\left(\tfrac{\chi^{(2)}(\tau)}{k\chi(\tau)}+\tfrac{1}{4\pi^2}\tfrac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}\right)\left(\theta_2^4(0,\tau)-\theta_4^4(0,\tau)\right)\ \Big\}\Big] \\
&\qquad\quad +\tfrac{i}{2}\chi^{(3)}(\tau)\left(f_e^{ab}d^{ecd}\Delta_1(\nu_{12}) + f_e^{ac}d^{bed}\Delta_1(\nu_{13}) + f_e^{ad}d^{bce}\Delta_1(\nu_{14})\right. \\
&\qquad\qquad\quad \left.+f_e^{bc}d^{aed}\Delta_1(\nu_{23}) + f_e^{bd}d^{ace}\Delta_1(\nu_{24}) + f_e^{cd}d^{abe}\Delta_1(\nu_{34})\right)\Big\}.
\end{aligned}
$$

We introduce traces over the group matrices to express the combinations of structure constants and d-symbols that occur in the four-point loop. For $\sigma^{abcd} \equiv \mathrm{tr}(T^aT^bT^cT^d)$,

$$
\begin{aligned}
\sigma^{abcd} - \sigma^{adcb} + \sigma^{acdb} - \sigma^{abdc} &= i f_e^{cd} d^{abe}, \\
-\sigma^{abcd} - \sigma^{adcb} + \sigma^{acdb} + \sigma^{abdc} &= 2k f_e^{ab} f^{cde}.
\end{aligned}
$$

The symmetrization of the trace $\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})$ over permutations of a, b, c, d is

$$
\begin{aligned}
&\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0}) \\
&= \left(\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})\right)_{\mathbf{S}} \\
&\quad +\tfrac{i}{4}\left(f_e^{cd}d_{abe} + f_e^{bd}d_{ace} + f_e^{bc}d_{ade}\right)\chi^{(3)}(\tau) + \tfrac{i}{6}\left(f_e^{bc}f_{ade} - f_e^{cd}f_{abe}\right)\chi^{(2)}(\tau).
\end{aligned}
$$

Then the four-point loop can be written in the symmetric form (6.22).

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